Math 2 Class-notes by Amalansu Sekhar Pattanayak for B.Sc (General) 2nd Sem students



Solution of Clairaut's Equation.

A differential equation of the form

$y = px + f(p) \rightarrow (1)$ is known as Clairaut's equation where $p = \frac{dy}{dx}$.

Now differentiating w.r.t $x$ the ~~the~~ Clairaut's ~~eq~~ equation (1) we have:

$\frac{dy}{dx} = p = p + \{x + f'(p)\frac{dp}{dx}\}$

or, $\frac{dp}{dx}\{x + f'(p)\} = 0$ Either $\frac{dp}{dx} = 0$ — (2)

integrating $p = c$ (arbitrary Const) or $\{x + f'(p)\} = 0 \Rightarrow (3)$

Now putting $p = c$ in the given eqn. $y = px + f(p)$ we get $y = cx + f(c)$ which is the general soln.

Case-II when $x + f'(p) = 0$ then eliminating $p$ from the given eqn (1) & eqn (3) we get another solution which ~~do not~~ does not contain $c$ (arbitrary Const) is called a singular solution which can not be obtained by giving any particular value of $c$ in general soln.

EX. Obtain the Complete primitive and the singular solution of the equation $y = px + \sqrt{1+p^2}$

Soln: This equation is in Clairaut's form. So differentiating w.r.t $x$ we get

Next

$y = px + \sqrt{1+p^2}$ ---- ①

diff. w. r. to $x$

$\frac{dy}{dx} = p = \cancel{p} + x\frac{dp}{dx} + \frac{p}{\sqrt{1+p^2}} \cdot \frac{dp}{dx}$

or, $\frac{dp}{dx}\left\{x + \frac{p}{\sqrt{1+p^2}}\right\} = 0$

Case ① $\frac{dp}{dx} = 0$ integrating

$\int dp = \int 0\, dx \Rightarrow p = c$ ---- ②

Case ⑪ $x + \frac{p}{\sqrt{1+p^2}} = 0$ or $x = \frac{-p}{\sqrt{1+p^2}}$ ... ③

~~elim~~ eliminating $p$ between equ$^n$ ① and ②

we get, if we put $p = c$ in ①

$y = cx + \sqrt{1+c^2}$ which general sol$^n$.

again eliminating $p$ between ① and ③ ~~we get~~

$\boxed{y = px - \frac{p}{\sqrt{1+p^2}} + \cancel{\sqrt{1+p^2}}}$ By squaring ① and equ$^n$ ③ and adding

we get, $x^2 + y^2 = \left(\frac{-p}{\sqrt{1+p^2}}\right)^2 + \left(px + \sqrt{1+p^2}\right)^2$

$= \frac{p^2}{1+p^2} + \left(p \times \frac{-p}{\sqrt{1+p^2}} + \sqrt{1+p^2}\right)^2$

$= \frac{p^2}{1+p^2} + \left(\frac{-p^2}{\sqrt{1+p^2}} + \sqrt{1+p^2}\right)^2 = \frac{p^2}{1+p^2} + \left(\frac{\cancel{-p^2} + 1 + \cancel{p^2}}{\sqrt{1+p^2}}\right)^2$

$= \frac{p^2}{1+p^2} + \frac{1}{1+p^2} = \frac{p^2+1}{1+p^2} = 1$

The required singular sol$^n$ is $x^2 + y^2 = 1$.

Ex ② Solve ~~the diff equⁿ Sin~~ the D.E. (diff. equⁿ)

$$\sin\left(x\frac{dy}{dx}\right)\cos y = \cos\left(x\frac{dy}{dx}\right)\sin y + \frac{dy}{dx}$$

Solution. Putting $p = \frac{dy}{dx}$ we get

$$\sin(xp)\cos y - \cos(xp)\sin y = \frac{dy}{dx} \qquad (\sin A\cos B - \cos A\sin B = \sin(A-B))$$

or, $\sin(xp - y) = p$ —— ①

∴ $xp - y = \sin^{-1}p$ —— ② or $y = xp - \sin^{-1}p$ which is in Clairaut's form. differentiating with r. to $x$, we get

$$\frac{dy}{dx} = p = x\frac{dp}{dx} + p - \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

or, $x\frac{dp}{dx} - \frac{dp}{dx}\cdot\frac{1}{\sqrt{1-p^2}} = 0 \Rightarrow \frac{dp}{dx}\left\{x - \frac{1}{\sqrt{1-p^2}}\right\} = 0$

Either $\frac{dp}{dx} = 0 \Rightarrow dp = 0$ integrating $p = c$ (arbitrary const)

Putting $p = c$ in ① ie $\sin(cx - y) = c$

i.e $cx - y = \sin^{-1}c \Rightarrow y = cx - \sin^{-1}c$ —— ③ is the general solⁿ.

or, $x - \frac{1}{\sqrt{1-p^2}} = 0 \Rightarrow x = \frac{1}{\sqrt{1-p^2}} \Rightarrow x^2(1-p^2) = 1$

or, $\cancel{x^2 = \frac{1}{1-p^2}} = x^2 - x^2p^2 = 1 \Rightarrow p^2x^2 = x^2 - 1$

or, $p^2 = \frac{x^2-1}{x^2} \Rightarrow p = \frac{\sqrt{x^2-1}}{x}$ —— ④

Now eliminating $p$ between ② and ④ we get

$$x \cdot \frac{\sqrt{x^2-1}}{x} - y = \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right) \Rightarrow y = \sqrt{x^2-1} - \sin^{-1}\frac{\sqrt{x^2-1}}{x}$$

is the singular solution.

Ex ③ use the transformation $u = x^2$ and $v = y^2$ to ~~to~~ to reduce the diff. Equ$^n$. $xyp^2 - (x^2+y^2-1)p + xy = 0 \rightarrow ①$ in Clairaut's equ$^n$ and then solve it.

Sol$^n$. Since $u = x^2$ and $v = y^2$ So $du = 2x\,dx$ and $dv = 2y\,dy$ therefore $\frac{dv}{du} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x} \cdot \frac{dy}{dx}$

Now put $q = \frac{dv}{du}$ then $\frac{y}{x} p = q$ $\quad \left(p = \frac{dy}{dx}\right)$

or $p = \frac{xq}{y}$ ~~Putting~~ putting the value of $p$ in equ$^n$ ①

we get $xy \frac{x^2q^2}{y^2} - (x^2+y^2-1)\frac{xq}{y} + xy = 0$ Dividing by $xy$

~~or~~ we get $\frac{x^2q^2}{y^2} - (x^2+y^2-1)\frac{q}{y^2} + 1 = 0$

or, $x^2q^2 - (x^2+y^2-1)q + y^2 = 0$ (Now putting $x^2 = u$ and $v^2 = u$)

$uq^2 - (u+v-1)q + v = 0 \Rightarrow uq^2 - uq - vq + q + v = 0$

or, $v(1-q) - uq(1-q) + q = 0$

or, $v(1-q) = uq(1-q) - q$

or, $v = uq + \frac{q}{q-1} \longrightarrow ②$

Here ~~y~~ as $v$ ~~x~~ as $u$ }

which is Clairaut's form.

(Like $y = px + f(p)$ here $v = qu + f(q)$)

Therefore the general solution will be obtained by putting $q = c$ in ② i.e. $v = uc + \frac{c}{c-1}$, $c$ is arbitrary constant, restoring the values of $u$ & $v$; we get the general solution as $y^2 = cx^2 + \frac{c}{c-1}$.

(Ex) do yourself ! Solve the diff. equ$^n$ by reducing to Clairaut's form by putting $x^2 = u$, and $y^2 = v$, the diff. Equ$^n$ $(px - y)(x - py) = 2p$.

Next Page-5

Ex ④ Reduce the equation $(px^2+y^2)(px+y) = (p+1)^2$ in Clairaut's form by using the substitution $u = xy$, and $v = x+y$ and find Complete primitive

Solⁿ: Putting $u = xy$ and $v = x+y$

$\therefore \frac{du}{dx} = y + x\frac{dy}{dx}$ $\qquad \frac{dv}{dx} = 1 + \frac{dy}{dx} = 1 + p$ $\quad \left(p = \frac{dy}{dx}\right)$

$= y + xp$

~~Now~~ ~~do~~ there for $\frac{y+xp}{p+1} = \frac{du}{dv} = q$ — ①

the given eqⁿ $(px^2+y^2)(px+y) = (1+p)^2$ becomes

(Now $px^2+y^2 = (px+y)(x+y) - xy(p+1)$)

so $\{(px+y)(x+y) - xy(p+1)\}(px+y) = (p+1)^2$

or $(px+y)^2(x+y) - xy(p+1)(px+y) = (p+1)^2$

$\left(\frac{px+y}{p+1}\right)^2(x+y) - \frac{(px+y)xy}{p+1} = 1$ dividing by $(p+1)^2$

or $q^2 v - qu = 1$ of using $u = xy$, $v = x+y$

or, $u = qv - \frac{1}{q} = qv - f(q)$ — ②

which is in Clairaut's form.

the diff w.r.t $v$ we get

$$\frac{du}{dv} = q + v\frac{dq}{dv} + \frac{1}{q^2}\cdot\frac{dq}{dv}$$

$$q = q + v\frac{dq}{dv} + \frac{1}{q^2}\frac{dq}{dv} \Rightarrow \frac{dq}{dv}\left(v + \frac{1}{q^2}\right) = 0$$

hence $\frac{dq}{dv} = 0 \Rightarrow dq = 0$ integrating $q = c$ (arbitrary Const.)

Putting $q = c$ in ② we get $u = cv - \frac{1}{c}$ Now ($u = xy$ & $v = x+y$)

Then the general primitive is $xy = c(x+y) - \frac{1}{c}$

or $cxy = c^2(x+y) - 1$ is the required solⁿ.

N2 END