Note 4 P. 1

Solution of non homogeneous linear diff. equation of $P_0\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_2y=R(x)$ — ① if $R(x)=0$ then Eqn ① is homogeneous.

For solution of $(P_0D^2+P_1D+P_2)y=R(x) \longrightarrow$ ② ~~has~~ has two parts First Eqn ② will be ~~transform~~ made homogeneous by putting $R(x)=0$ and there will be found C.F. (Complementary ~~function and the 2nd~~ function which is already known and 2nd part is Particular integral which has no arbitrary Const. So the general soln is $y=y_c+y_p$

y_c = complementary function and y_p = Particular integral (P.I.)

Rule for finding P.I (Particular integral.

$$P.I=\frac{1}{f(D)}R(x) \quad \text{where } f(D)=P_0D^2+P_1D+P_2 \text{ from ②}$$

function symbol D, means derivative, D^2 means 2nd order derivative ie $D(4x^2)=8x$. $D^2(4x^2)=D(8x)=8$

and $\frac{1}{D}$ means integration ie $\frac{1}{D}(x^2)=\frac{x^3}{3}$, $\frac{1}{D^2}(x)=\frac{1}{D}\left(\frac{x^2}{2}\right)=\frac{x^3}{6}$ etc

To find P.I. of $\frac{1}{f(D)}\cdot e^{ax}=e^{ax}\cdot\frac{1}{f(a)}$ if $f(a)\neq 0$.

if $f(a)=0$, the Rule fails. Let $f(D)=(D-a)^n$, $n=1,2,\ldots$

then $P.I=\frac{1}{(D-a)^n}\cdot e^{ax}=e^{ax}\cdot\frac{1}{(D+a-a)^n}\cdot 1=e^{ax}\cdot\frac{1}{D^n}\cdot 1$

again if $f(D)=(D-a)^r\phi(D)$, then

$$P.I=\frac{1}{(D-a)^r\phi(D)}e^{ax}=\frac{1}{(D-a)^r}\cdot\frac{1}{\phi(D)}\cdot e^{ax}=\frac{1}{\phi(a)}\cdot\frac{e^{ax}}{(D-a)^r}$$

$$=\frac{1}{\phi(a)}\cdot e^{ax}\cdot\frac{x^r}{r!} \quad \text{Provided } \phi(a)\neq 0.$$

To find the P.I. of $\frac{1}{f(D^2)} \sin ax = \frac{\sin ax}{f(-a^2)}$ Put $D^2 = -a^2$ if $f(-a^2) \neq 0$.

if $f(-a^2) = 0$, then $\frac{1}{f(D^2)} \sin ax = x \cdot \frac{\sin ax}{f'(-a^2)}$ if $f'(-a^2) \neq 0$

if $f'(-a^2) = 0$ then $\frac{1}{f(D^2)} \sin ax = x^2 \cdot \frac{\sin ax}{f''(-a^2)}$

if $f''(-a^2) \neq 0$

etc

Consult Diff. Eqn by Chakravarty & Ghosh.

Find P.I. of $\frac{1}{D^2+9}(x \sin x)$ = imaginary part (I.P.) of $\frac{1}{D^2+9} x e^{ix}$

$= \cancel{\text{I.P. of } \frac{1}{D^2+9}(x \sin x)} = \text{I.P. of } e^{ix} \frac{1}{(D+i)^2+9} \cdot x$

$\left[\text{By Rule } \frac{1}{f(D)} \cdot v \cdot e^{ax} = \frac{1}{f(D+a)} \cdot v\right]$

$= \text{I.P. of } e^{ix} \frac{1}{D^2+2iD+8} \cdot x = \text{I.P. of } e^{ix} \cdot \frac{1}{8\left\{1 + \frac{D^2+2iD}{8}\right\}} \cdot x$

$= \text{I.P. of } \frac{e^{ix}}{8}\left[1 + \frac{iD}{4} + \frac{D^2}{8}\right]^{-1} x$

$= \text{I.P. of } \frac{e^{ix}}{8}\left[1 - \frac{iD}{4} + \cdots\right] x$ as $D^2(x) = 0$ neglecting D^2 & higher power of D.

$= \text{I.P. of } \frac{1}{8}(\cos x + i \sin x)\left(x - i \cdot \frac{1}{4}\right)$

$= \text{imaginary part of } \left\{\frac{1}{8} x \cos x - \frac{1}{32} i \cos x + \frac{1}{8} x i \sin x + \frac{1}{32} \sin x\right)$

$= \frac{1}{8}\left(x \sin x - \frac{1}{4} \cos x\right)$ is the required result.

$\frac{N}{4}$ / P_3 3

Ex ① Solve the diff. Eq^n $(D^2-2D+2)y = 5\cos x$ given that $y=0$ at $\frac{dy}{dx}=0$, when $x=0$. → ①

Sol^n: The given diff. eq^n ① is non homogeneous. its sol^n is $y = y_c + y_p$.

To find C.F we shall solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ → ②

Let $y = e^{mx} \neq 0$ be a trial solution of eq^n ②

Then A.E is $e^{mx}(m^2-2m+2)=0$ putting values $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$, y in ②

as $e^{mx} \neq 0$ so $m^2-2m+2=0$

$\therefore m = \frac{2 \pm \sqrt{4-4.2}}{2.1} = \frac{2 \pm 2i}{2} = 1 \pm i$

The roots ~~are~~ of A.E is complex conjugate so

C.F $(y_c) = e^x(c_1\cos x + c_2\sin x)$, c_1, c_2 arbitrary consts.

Fo P.I we write the eq^n ① as $(D^2-2D+2)y = 5\cos x$

$\therefore y_p = \text{P.I.} = \frac{1}{D^2-2D+2} 5\cos x = 5.\frac{1}{(-1)^2-2D+2}.\cos x$

(putting $D^2 = -1^2$)

$y_p = \text{P.I} = 5.\frac{1}{1-2D}.\cos x = 5.\frac{(1+2D)}{1-4D^2}.\cos x$

$= 5\frac{(1+2D)}{1-(4)(-1)}\cos x$ Now put $D^2 = -1^2$

$= \frac{5}{5}(\cos x + 2D\cos x) = (\cos x - 2\sin x)$.

Hence general sol^n $y = y_c + y_p$

$y = e^x(c_1\cos x + c_2\sin x) + \cos x - 2\sin x$ ③

~~required sol~~^{n}. to find c_1, c_2 use given condition

$\therefore \frac{dy}{dx} = (c_1\cos x + c_2\sin x)e^x + (-c_1\sin x + c_2\cos x)e^x - \sin x - 2\cos x$ ④

put $y=0$, $x=0$ in ③ we get $0 = c_1+1 \Rightarrow c_1 = -1$

Now $\frac{dy}{dx}=0$, $x=0$ in ④ we get $0 = c_1+c_2-2 \Rightarrow c_2 = 2-c_1 = 3$ as $c_1=-1$

Hence the required sol^n is

$y = (3\sin x - \cos x)e^x + \cos x - 2\sin x$.

Solution of nonhomogeneous diff. equ^n. ——①

Ex ② Solve $\frac{d^2y}{dx^2}+4y=\sin 2x$

Sol^n. To find C.F Let solve the equ^n $(D^2+4)y=0$ — ②

if $y=e^{mx}$ be a trial sol^n of Equ^n ② then $\frac{dy}{dx}=me^{mx}$

$\frac{d^2y}{dx^2}=m^2e^{mx}$ so A.E. will be obtained on putting the value of Dy & D^2y in ② ie $e^{mx}(m^2+4)=0$, $e^{mx}\neq 0$

so $m^2+4=0 \Rightarrow m=\pm 2i = 0\pm 2i$

Roots are complex so $C.F\ (y_c)=e^{0x}(A\cos 2x+B\sin 2x)$

where A, B are arbitrary constants

To P.I. (y_p) we write $y_p=\frac{1}{D^2+4}\sin 2x$

Here $D^2=-2^2$ becomes $f(-a^2)=0$ in $f(D^2)=D^2+4$

So $y_p=$ imaginary part of $\frac{1}{D^2+4}e^{2ix}$

$=$ I.P. of $e^{2ix}\frac{1}{(D+2i)^2+4}\cdot 1$

$[e^{2ix}=\cos 2x+i\sin 2x]$

$=$ I.P. of $e^{2ix}\frac{1}{D^2+4iD+4i^2+4}\cdot 1$ $\quad i^2=-1$

$=$ I.P. of $e^{2ix}\cdot\frac{1}{4iD\left(1+\frac{D}{4i}\right)}\cdot 1$

$=$ I.P. of $e^{2ix}\frac{1}{4iD}\left(1+\frac{D}{4i}\right)^{-1}\cdot 1$

$=$ I.P. of $e^{2ix}\frac{1}{4iD}\left(1-\frac{D}{4i}+\cdots\right)\cdot 1$ neglecting D^2, because $D^2(1)=0$

$=$ I.P. of $e^{2ix}\frac{1}{4iD}(1-0)\cdot 1$

$=$ I.P. of $(\cos 2x+i\sin 2x)\frac{i}{4i^2}\frac{1}{D}\cdot 1$

$=$ I.P. of $(\cos 2x+i\sin 2x)\times\frac{i}{-4}\cdot x$

$=-\frac{1}{4}x\cos 2x$

Hence general sol^n is $y=(A\cos 2x+B\sin 2x)-\frac{x}{4}\cos 2x$

EX ③ Solve $(D^2-2D+1)y = e^{3x}\cos x$

Sol^n: To find C.F Let $(D^2-2D+1)y=0$

By a trial sol^n $y=e^{mx}$ the A.E

is $e^{mx}(m^2-2m+1)=0$, as $e^{mx}\neq 0$ so

$m^2-2m+1=0 \Rightarrow m=1,1$ equal roots

so $y_c = (c_1+c_2x)e^{1x} = (c_1+c_2x)e^{x}$

Now to find P.I (Particular integral)

$y_p = \frac{1}{(D-1)^2}e^{3x}\cos x$

$= e^{3x}\frac{1}{(D+3-1)^2}\cdot\cos x$

$= e^{3x}\cdot\frac{1}{(D+2)^2}\cdot\cos x$

$= e^{3x}\cdot\frac{1}{D^2+4D+4}\cdot\cos x$

By Rule $\frac{1}{f(D)}e^{ax}\cdot V$

$= e^{ax}\cdot\frac{1}{f(D+a)}\cdot V.$

Put $D^2=-1^2$

$= e^{3x}\cdot\frac{1}{-1+4D+4}\cdot\cos x$

$= e^{3x}\cdot\frac{1}{4D+3}\cdot\cos x$

$= e^{3x}\frac{4D-3}{16D^2-9}\cdot\cos x$

Put $D^2=-$

$= e^{3x}\cdot\frac{1}{16(-1^2)-9}(4D-3)\cos x$

$= e^{3x}\times\frac{1}{-25}\cdot(-4\sin x-3\cos x)$

$= \frac{e^{3x}}{25}(4\sin x+3\cos x)$

Hence Complete Primitive is

$y=(c_1+c_2x)e^{x}+\frac{e^{3x}}{25}(4\sin x+3\cos x)$

EX. Solve $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y = x^2e^{x}.$