Semester- VI

Course Type - DSE-4

Course Title - DSE-4 : Mathematics Modeling

Topic: Laplace Transformation

References: Dr. Arup Mukherjee & Dr. Naba Kumar Bej Book.

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Now, using the convolution theorem, we get

$$L^{-1}\{f(s)g(s)\} = \int_0^t G(u)\, F(t-u)du$$

or, $$L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = \int_0^t \sin u . e^{-(t-u)}du \quad \text{[using (1) and (2)]}$$

$$= e^{-t}\int_0^t e^u \sin u du$$

$$= e^{-t}\left[\frac{e^u}{1^2+1^2}(\sin u - \cos u)\right]_0^t$$

$$= \frac{e^{-t}}{2}\left[e^t(\sin t - \cos t) - e^0(0-1)\right]$$

$$= \frac{1}{2}[\sin t - \cos t + e^{-t}]$$

Example 14. Use convolution theorem to show that $L^{-1}\left\{\frac{1}{(p+2)^2(p-2)}\right\} = \frac{1}{16}(e^{2t} - 4te^{-2t} - e^{-2t})$. **[C.U. (Hons.) 2006]**

Solution :

Let $f(p) = \frac{1}{p-2}$ and $g(p) = \frac{1}{(p+2)^2}$

Then $$F(t) = L^{-1}\{f(p)\} = L^{-1}\left(\frac{1}{p-2}\right) = e^{2t}$$

and $$G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{(p+2)^2}\right\}$$

$$= e^{-2t}L^{-1}\left(\frac{1}{p^2}\right), \text{ using first shifting theorem.}$$

$$= t\,e^{-2t}$$

Now, by the convolution theorem, we have

$$L^{-1}\{f(p)g(p)\} = \int_0^t G(u)\,F(t-u)\,du$$

So, $$L^{-1}\left\{\frac{1}{(p-2)(p+2)^2}\right\} = \int_0^t ue^{-2u}\,e^{2(t-u)}du = e^{2t}\int_0^t ue^{-4u}du$$

$$= e^{2t}\left[-u\frac{e^{-4u}}{4}\Big|_0^t + \int_0^t 1.\frac{1}{4}e^{-4u}du\right]$$

$$= e^{2t}\left(-\frac{t}{4}e^{-4t} - \frac{1}{16}e^{-4t} + \frac{1}{16}\right)$$

$$= \frac{1}{16}(e^{2t} - 4te^{2t} - e^{-2t}) \text{ **Proved.**}$$

Example 15. **Applying Laplace transform integrate the following** $\int_0^\infty \frac{e^{-3t} - e^{6t}}{t} dt$.

Solution :

We have,

$$L\{e^{-3t} - e^{-6t}\} = L\{e^{-3t}\} - L\{e^{-6t}\}$$

$$= \frac{1}{s+3} - \frac{1}{s+6} = f(s) \text{ (say)}$$

$$\therefore \quad L\left\{\frac{e^{-3t} - e^{-6t}}{t}\right\} = \int_s^\infty f(s)ds = \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+6}\right) ds$$

$$= \left[\log \frac{s+3}{s+6}\right]_s^\infty = \lim_{s\to\infty} \frac{s+3}{s+6} - \log \frac{s+3}{s+6} = \log \frac{s+6}{s+3}$$

$$\therefore \quad \int_0^\infty e^{-st}\left(\frac{e^{-3t} - e^{-6t}}{t}\right) dt = \log \frac{s+6}{s+3}, \text{ by definition}$$

Taking limit on both sides as $s \to 0$ we get

$$\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t} dt = \log \frac{6}{3} = \log 2$$

Example 16. Show that $\int_0^\infty e^{-t} \frac{\sin t}{t} dt = \frac{\pi}{4}$.

Solution :

We know

$$L\{\sin t\} = \frac{1}{s^2+1} = f(s) \text{ say}$$

So,

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty f(s)ds = \int_s^\infty \frac{1}{s^2+1} ds$$

$$= \left[\tan^{-1} s\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s$$

or, $$\int_0^\infty e^{-st} \frac{\sin t}{t} dt = \frac{\pi}{2} - \tan^{-1}s, \text{ by defination.}$$

Taking limit of both sides of as $s \to 1$, we get

$$\int_0^\infty e^{-t} \frac{\sin t}{t} dt = \frac{\pi}{2} - \tan^{-1}1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Example 17. Prove that $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt = \log\left(\frac{2}{3}\right)$.

Solution :

We know $L\{\cos 6t - \cos 4t\} = L\{\cos 6t\} - L\{\cos 4t\} = \frac{s}{s^2+6^2} - \frac{s}{s^2+4^2}$

$$= \frac{s}{s^2+36} - \frac{s}{s^2+16} = f(s) \text{ say,}$$

$$\therefore \quad L\left\{\frac{\cos 6t - \cos 4t}{t}\right\} = \int_s^\infty f(s)ds = \int_0^\infty \left(\frac{s}{s^2+36} - \frac{s}{s^2+16}\right)ds$$

$$= \frac{1}{2}\left[\log\frac{s^2+36}{s^2+16}\right]_s^\infty$$

$$= \frac{1}{2}\lim_{s\to 0}\log\frac{s^2+36}{s^2+16} - \frac{1}{2}\log\frac{s^2+36}{s^2+16}$$

$$= \frac{1}{2}\log\frac{s^2+36}{s^2+16}$$

or, $\int_0^\infty e^{-st}\frac{\cos 6t - \cos 4t}{t}dt = \frac{1}{2}\log\frac{s^2+16}{s^2+36}$, by defination

Taking limit on both sides as $s \to 0$, we get

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t}dt = \frac{1}{2}\log\frac{16}{36} = \log\left(\frac{2}{3}\right)$$

EXERCISE 19(A)

- **Find the Laplace transform of the followings :**

1. (i) $L\{1 - 2t^2\}$ (ii) $L\{2t^2 - 5t + 3\}$
2. $L\{3e^{2t} + 4e^{-3t}\}$
3. $L\left\{\frac{e^{kt}-1}{k}\right\}$
4. $L\{\sin^2 at\}$
5. $L\{\cos^3 t\}$
6. $L\{6\sin 2t - 5\cos 2t\}$
7. $L\{\cos t(\sin t + \cos t)\}$
8. $L\{7e^{2t} + 9e^{-2t} + 5\cos t + 7t^3 + 5\sin 3t + 2\}$
9. $L\{F(t)\}$, if $F(t) = 1, 0 < t < 2$
   $= t, t > 2$
10. $L\{H(t)\}$, if $H(t) = \sin 2t, 0 < t < \pi$
    $= 0, t > \pi$
11. $L\{F(t)\}$, where $F(t) = 0$, if $0 < t < 1$
    $= t$, if $1 < t < 2$
    $= 0$, if $t > 2$

12. Prove that $L\{\sin at \cos at\} = \dfrac{a}{s^2 + 4a^2}$, $s > 0$

13. Show that $L\{F(t)\} = \dfrac{2}{s^2}(1 - e^{-5s}) - \dfrac{9}{s}e^{-5s}$, $s > 0$

where $F(t) = 2t,\ 0 \le t \le 5$
$= 1,\ t > 5$

14. Use shifting theorem to show that

$$L\{(1 + te^{-t})^3\} = \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^2} + \frac{6}{(s+3)^4}$$

15. Use shifting theorem evaluate $L\left\{\dfrac{e^{-at}t^{n-1}}{(n-1)!}\right\}$

16. Use shifting theorem find the value of $L\{e^{at}\cos bt\}$

17. (i) Applying change of scale property, prove that, if

$$L\{F(t)\} = \frac{s^2 - s + 1}{(2s+1)^2(s-1)},\ \text{then } L\{F(2t)\} = \frac{s^2 - 2s + 4}{4(s+1)^2(s-2)}$$

(ii) Applying change of scale property and shifting theorem prove that,

$$L\{e^{-t}F(3t)\} = \frac{1}{s+1}e^{-\frac{3}{s+1}},\ \text{if } L\{F(t)\} = \frac{e^{-\frac{1}{s}}}{s}.$$

18. Evaluate $L\{t^n e^{at}\}$

19. Evaluate $L\{(at^2 + bt + c)e^{kt}\}$

20. Show that $L\left\{\dfrac{1 - e^t}{t}\right\} = \log\dfrac{s-1}{s}$

21. Prove that $L\left\{\dfrac{\cos at - \cos bt}{t}\right\} = \dfrac{1}{2}\log\dfrac{s^2 + b^2}{s^2 + a^2}$

22. Evaluate $L\{t^2\cos at\}$

23. Evaluate $L\{te^{-t}\sin t\}$

24. Prove that $L\left\{\dfrac{\sin^2 t}{t}\right\} = \dfrac{1}{4}\log\dfrac{s^2 + 4}{s^2}$

25. Find $L^{-1}\left\{\dfrac{s^2}{(S+1)(s+2)(s+3)}\right\}$

26. Evaluate $L^{-1}\left\{\dfrac{1 + 2s}{(s+2)^2(s-1)^2}\right\}$

27. Evaluate $L^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$

28. Find $L^{-1}\left\{\dfrac{5s + 3}{(s-1)(s^2 + 2s + 5)}\right\}$

29. If $L^{-1}\left\{\dfrac{s}{(s+1)^2}\right\} = \dfrac{t}{2}\sin t$, then find $L^{-1}\left\{\dfrac{32s}{(16s^2+1)^2}\right\}$

30. Evaluate $L^{-1}\left\{\log\dfrac{1+s}{s}\right\}$

31. Use convolution theorem evaluate

(i) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$, (ii) $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\}$, (iii) $L^{-1}\left\{\frac{1}{\sqrt{s}(s-1)}\right\}$.

32. Evaluate $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt$

33. Prove that $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}dt = \log\left(\frac{2}{3}\right)$

34. Evaluate $\int_0^\infty t^3 e^{-t}\sin t\, dt$

35. Prove that $\int_0^\infty \frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}$

Answers

1. (i) $\frac{s^2-4}{s^3}$; (ii) $\frac{1}{s^3}(4-5s+3s^2)$

2. $\frac{7s+1}{s^2+s-6}$, $s>2$

3. $\frac{1}{s(s-k)}$ if $s>k$

4. $\frac{2a^2}{s(s^2+4a^2)}$, $s>0$

5. $\frac{s(s^2+7)}{(s^2+1)(s^2+9)}$

6. $\frac{12-5s}{s^2+4}$, $s>0$

7. $\frac{s^2+s+2}{s(s^2+4)}$; $s>0$

8. $\frac{4(4s-1)}{s^2-4}+\frac{5s}{s^2+1}+\frac{15}{s^2+9}+\frac{2(s^3+21)}{s^4}$

9. $\frac{1}{s}(1+e^{-2s})+\frac{e^{-2s}}{s^2}$, $s>0$

10. $\frac{2(1-e^{\pi s})}{s^2+4}$

11. $\frac{e^{-s}}{s^2}\{(s+1)-(2s+1)e^{-s}\}$

15. $\frac{1}{(s+a)^n}$

16. $\frac{s-a}{(s-a)^2+b^2}$

18. $\frac{n!}{(s-a)^{n+1}}$, $s>a$

19. $\frac{-2a}{(s-k)^3}+\frac{b}{(s-k)^2}+\frac{c}{s-k}$, $s>k$

22. $\frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$

23. $\frac{2(s+1)}{(s^2+2s+2)^2}$

25. $\frac{1}{2}e^{-t}-4e^{-2t}+\frac{9}{2}e^{-3t}$

26. $\frac{t}{3}(e^t-e^{-2t})$

27. $2e^t+\sin t-2\cos t$

28. $e^t-e^{-t}(\cos 2t-\frac{3}{2}\sin 2t)$

29. $\frac{t}{4}\sin\frac{t}{4}$

30. $\frac{1-e^{-t}}{t}$

31. (i) $\frac{t\sin at}{2a}$ (ii) $\frac{1}{2a}(at\cos at+\sin at)$ (iii) $e^t erf(\sqrt{t})$

32. $\log 3$

33. $\log\frac{2}{3}$

34. 0.

19.14 Laplace Transform of Derivatives

Theorem : Let the function $F(t)$ be continuous with a sectionally (piece-wise) continuous derivative $F'(t)$ in every finite interval $0 \le t \le T$. Also let $F(t)$ be of order $e^{\alpha t}$ as $t \to \infty$. Then when, $s > \alpha$, the transformation of $F'(t)$ exists and

$$L\{F'(t)\} = sL\{F(t)\} - F(0).$$

Proof : Since $F(t)$ is continuous at $t = 0$, $F(+0)$ is the same as $F(0)$. Now to prove the theorem, we first note that

$$L\{F'(t)\} = \underset{T\to\infty}{Lt} \int_0^T e^{-st} F'(t)dt,$$

if this limit exists. We write the integral here as the sum of the integrals over subintervals, in each of which the integral is continuous. For any given T, let $t_1, t_2 \ldots t_n$ denote the values of t in $(0, T)$ for which $F'(t)$ is discontinuous. Then

$$\int_0^T e^{-st} F'(t)dt = \int_0^{t_1} e^{-st} F'(t)dt + \int_{t_1}^{t_2} e^{-st} F'(t)dt + \ldots + \int_{t_n}^{T} e^{-st} F'(t)dt$$

$$= e^{-st_1} F(t_1) - F(0) + s\int_0^{t_1} e^{-st} F(t)dt + e^{-st_2} F(t_2) - e^{-st_1} F(t_1) +$$

$$s\int_{t_1}^{t_2} e^{-st} F(t)dt + \ldots + e^{-sT}F(T) - e^{-st_n} F(t_n) + s\int_{t_n}^{T} e^{-st} F(t)dt$$

[Integrating by parts.]

$$= -F(0) + e^{-sT}F(T) + s\left[\int_0^{t_1} e^{-st}F(t)dt + \int_{t_1}^{t_2} e^{-st}F(t)dt + \ldots + \int_{t_n}^{T} e^{-st}F(t)dt\right]$$

$$= -F(0) + e^{-sT}F(T) + s\int_0^T e^{-st}F(t)dt \quad \ldots (1)$$

Since we can choose T in such a way that

$$\left|e^{-sT}F(T)\right| < Me^{-(s-\alpha)T}$$

for some positive T and since $s > \alpha$, $e^{-sT}F(T)$ vanishes as $T \to \infty$. Also the last integral in equation (1) approaches $L\{F(t)\}$ as $T \to \infty$. Thus the right hand side of (1) has a limit as $t \to \infty$ and left hand side of (1) also has a limit as $t \to \infty$. Thus for $s > \alpha$ $L\{F'(t)\}$ exists and is given by

$$L\{F'(t)\} = sL\{F(t)\} - F(0)$$

which proves the theorem.

Note : To obtain the transform of the derivatives of the second order $F''(t)$, we apply the above theorem to the function $F'(t)$. Let $F'(t)$ be continuous and $F''(t)$ be sectionally continuous in each finite interval and let $F(t)$ and $F'(t)$ be of exponential order, then

$$L\{F''(t)\} = sL\{F'(t)\} - F'(0) = s[sL\{F(t)\} - F(0)] - F'(0)$$
$$= s^2L\{F(t)\} - sF(0) - F'(0)$$
$$= s^2f(s)\} - sF(0) - F'(0)$$

By successive application of the above theorem, we may write

$$L\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \ldots - sF^{n-2}(0) - F^{n-1}(0)$$

19.15 Solution of Ordinary Differential Equations by Laplace Transform

Let us consider the second order differential equation

$$a_0 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_2 x = f(t)$$

$$\text{or,} \quad (a_0 D^2 + a_1 D + a_2)x = f(t),\ D \equiv \frac{d}{dt} \qquad \dots (1)$$

where a_0, a_1, a_2 are constants and which satisfy the initial conditions $x = x_0$, $\frac{dx}{dt} = x_1$, when $t = 0$

Taking Laplace transforms of both sides of (1), we get

$$a_0[s^2\bar{x} - sx_0 - x_1] + a_1[s\bar{x} - x_0] + a_2\bar{x} = \bar{f}(s)$$

$$\text{or,} \quad (a_0 s^2 + a_1 s + a_2)\bar{x} = a_0(sx_0 + x_1) + a_1 x_0 + \bar{f}(s) \qquad \dots (2)$$

Here $\bar{x}$, written for $\bar{x}(s)$, and $\bar{f}(s)$ stand for the Laplace transforms of $x(t)$ and $f(t)$ respectively. The equation (2) is called the subsidiary equation corresponding to the given differential equation (1).

From (2) we get

$$\bar{x} = \frac{a_0(sx_0 + x_1) + a_1 x_0 + \bar{f}(s)}{a_0 s^2 + a_1 s + a_2}$$

Splitting up the right hand side into partial fractions, we can find the expression for $x(t)$ by inverse Laplace transformation.

In case of boundary value problem, the method of procedure is somewhat modified as will be illustrated by examples. Laplace transform can also be used to some ordinary differential equation with variable co-efficients. When the terms of the differential equations is of the form $t^n y^n(t)$, then its Laplace transform is $(-1)^n \frac{d^n}{ds^n} L\{y^n(t)\}$.

Laplace transform can also be used to solve simultaneous differential equations.

19.16 Worked Out Example

Example 1. Solve the boundary value problem $y'' + 2y' + y = 0$, given $y(0) = 0$ and $y(1) = 2$, when $y'' = \frac{d^2y}{dt^2}$ and $y' = \frac{dy}{dt}$. **[C.U. (Hons.) 2006]**

Solution :

Given $\qquad y'' + 2y' + y = 0, \qquad \dots (1)$

with boundary condition $y(0) = 0$ and $y(1) = 2$

Taking Laplace transform of both sides of (1), we get

$$s^2L(y) - sy(0) - y'(0) + 2[sL(y) - y(0)] + L(y) = 0 \qquad \dots (2)$$

Here $y(0) = 0$ and $y'(0)$ is not known. Let $y'(0) = a$, a being a constant.

Then from (2) we get

$$L\{y\} = \frac{a}{(s+1)^2}$$

Taking inverse Laplace transform of both sides, we get

$$y(t) = ate^{-t}$$

Now apply the boundary condition $y(1) = 2$ we get $a = 2e$

Hence the required solution is $y = 2ete^{-t}$.

Example 2. Solve $(D^2 + 1)y = 6\cos 2t$, if $y = 3$, $Dy = 1$ when $t = 0$.

Solution :

The equation can be written as

$$y'' + y = 6\cos 2t, \quad \text{... (1)}$$

with the conditions $y(0) = 3$ and $y'(0) = 1$, where $y'' = D^2y$

Taking Laplace transform on the both sides of (1), we get

$$L\{y''\} + L\{y\} = 6L\{\cos 2t\}$$

$$\text{or,} \quad s^2L\{y\} - sy(0) - y'(0) + L\{y\} = \frac{6s}{s^2+4}$$

$$\text{or,} \quad (s^2 + 1)\, L\{y\} - 3s - 1 = \frac{6s}{s^2+4}$$

$$\text{or,} \quad L\{y\} = \frac{6s}{(s^2+4)(s^2+1)} + \frac{3s+1}{s^2+1}$$

$$\text{or,} \quad L\{y\} = 2s\left\{\frac{1}{s^2+1} - \frac{1}{s^2+4}\right\} + \frac{3s}{s^2+1} + \frac{1}{s^2+1}$$

$$\text{or,} \quad L\{y\} = \frac{5s}{s^2+1} - \frac{2s}{s^2+4} + \frac{1}{s^2+1}$$

Hence,
$$y = 5L^{-1}\left(\frac{s}{s^2+1}\right) - 2L^{-1}\left(\frac{s}{s^2+4}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= 5\cos t - 2\cos 2t + \sin t$$

Therefore the required solution is $y = 5\cos t - 2\cos 2t + \sin t$

Example 3. Solve $(D^2 + 2D + 5)y = e^{-t}\sin t$, $y(0) = 0$, $y'(0) = 1$.
[Delhi (Hons.) 1997]

Solution :

The equation is

$$y'' + 2y' + 5y = e^{-t}\sin t \quad \text{... (1)}$$

with initial conditions $y(0) = 0$, $y'(0) = 1$

Taking Laplace transform on both sides of (1), we get

$$L\{y''\} + 2L\{y'\} + 5L\{y\} = L\{e^{-t}\sin t\}$$

$$\text{or,} \quad s^2L\{y\} - sy(0) - y'(0) + 2\{sL(y) - y(0)\} + 5L\{y\} = \frac{1}{(s+1)^2+1}$$

or, $(s^2 + 2s + 5)\, L\{y\} - 1 = \dfrac{1}{(s+1)^2+1}$

or, $L\{y\} = \dfrac{1}{(s^2+2s+5)} + \dfrac{1}{\{(s+1)^2+1\}\{s^2+2s+5\}}$

or, $L\{y\} = \dfrac{1}{(s+1)^2+4} + \dfrac{1}{\{(s+1)^2+4\}\{(s+1)^2+1\}}$

or, $y = L^{-1}\left\{\dfrac{1}{(s+1)^2+4} + \dfrac{1}{\{(s+1)^2+4\}\{(s+1)^2+1\}}\right\}$

$= e^{-t}L^{-1}\left\{\dfrac{1}{s^2+4} + \dfrac{1}{(s^2+4)(s^2+1)}\right\}$, using shifting theorem

$= e^{-t}L^{-1}\left\{\dfrac{1}{s^2+4} + \dfrac{1}{3}\left(\dfrac{1}{s^2+1} - \dfrac{1}{s^2+4}\right)\right\}$

$= e^{-t}L^{-1}\left\{\dfrac{2}{3(s^2+4)} + \dfrac{1}{3}\cdot\dfrac{1}{s^2+1}\right\}$

$= e^{-t}\left[\dfrac{2}{3}\cdot\dfrac{1}{2}\sin 2t + \dfrac{1}{3}\sin t\right] = \dfrac{1}{3}e^{-t}(\sin 2t + \sin t)$

Hence $\quad y = \dfrac{1}{3}e^{-t}(\sin 2t + \sin t)$

which is the required solution.

Example 4. Solve $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = e^{-t}$, **given $y(0) = y'(0) = 0$.**

Solution :

The given equation may be taken as

$$y'' + 3y' + 2y = e^{-t} \quad \text{... (1)}$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$

Taking Laplace transform on both sides of (1) we get

$L\{y''\} + 3L\{y'\} + 2L\{y\} = L\{e^{-t}\}$

or, $s^2L\{y\} - sy(0) - y'(0) + 3[sL\{y\} - y(0)] + 2L\{y\} = \dfrac{1}{s+1}$

or, $(s^2 + 3s + 2)\, L\{y\} = \dfrac{1}{s+1}$

or, $L\{y\} = \dfrac{1}{(s+1)(s^2+3s+2)} = \dfrac{1}{(s+2)(s+1)^2}$

or, $L\{y\} = \dfrac{1}{s+2} - \dfrac{1}{s+1} + \dfrac{1}{(s+1)^2}$