Semester-IV

Course Type - Core-8

Course Title - C8T: Sequence of function

Topic: Some Examples of sequence of function

References: S.K. Mapa Book.

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This proves that $\{f_n\}$ is uniformly convergent on $[0,\infty)$.

Second Part : Now, $f'_n(x) = \dfrac{1+nx-nx}{(1+nx)^2} = \dfrac{1}{(1+nx)^2}$, $x \geq 0$.

$\because f'_n(0) = 1, \forall n \in N$

$\therefore \lim_{n\to\infty} f'_n(0) = 1$. Also, $f'(0) = 0$.

Thus, we have $\lim_{n\to\infty} f'_n(0) \neq f'(0)$.

Ex. Let $\{f_n\}$ be a sequence of functions on an interval I that converges uniformly on I to a continuous function f. Let $c \in I$ and $\{x_n\}$ is any sequence in I converges to c.

Prove that $\lim_{n\to\infty} f_n(x_n) = f(c)$.

Solution : Let $\in > 0$. Since $\{f_n\}$ is uniformly convergent on I, then $\exists$ a natural number k_1, such that $\forall x \in I$, $|f_n(x) - f(x)| < \dfrac{\in}{2}$, $\forall n \geq k_1$ ----------(1)

Since f is continuous at $c \in I$, $\exists$ a positive δ such that,

$$|f(x) - f(c)| < \frac{\in}{2}, \quad |x-c| < \delta \text{ ---------- (2)}$$

Since, $\lim_{n\to\infty} x_n = c$, $\exists$ a natural number k_2 such that

$$|x_n - c| < \delta, \ \forall n \geq k_2 \text{ ---------- (3)}$$

Let $k = \max\{k_1, k_2\}$.

Then, from (1), $|f_n(x_n) - f(x_n)| < \dfrac{\in}{2}$, $\forall n \geq k$ ---------- (4)

From (2) using (3), we have

$$|f(x_n) - f(c)| < \frac{\in}{2}, \ \forall n \geq k \text{ ---------- (5)}$$

Now, $|f_n(x_n) - f(c)| = |f_n(x_n) - f(x_n) + f(x_n) - f(c)|$

$\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(c)|$

$< \dfrac{\in}{2} + \dfrac{\in}{2} = \in$, $\forall n \geq k$, using (4) and (5).

$\Rightarrow \lim_{n\to\infty} f_n(x_n) = f(c)$.

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Ex: Let $\{f_n\}$ be a sequence of differentiable function on [a, b] that converges on [a, b] to f. If $\{f_n'\}$ be a sequence of continuous function on [a, b] that converges uniformly to g on [a, b] then show that $g(x) = f'(x),\ x \in [a, b]$.

Solution : Since $\{f_n'\}$ is a uniformly convergent sequence of continuous functions on [a, b], then the limit function g is continuous on [a, b].

Since, each f'_n is continuous on [a, b].

$\therefore$ Each f'_n is integrable on [a, b].

Since, $\{f_n'\}$ is uniformly convergent on [a, b] to the limit function g.

Then, we have

2nd: Since $\{f_n'\} \xrightarrow{\text{uniform}} g$ Then $\{f_n\}$ is also uniform convergent.

$$\forall x \in [a,b],\ \lim_{n\to\infty} \int_a^x f_n'(t)dt = \int_a^x g(t)dt.$$

But, $\int_a^x f_n'(t)dt = f_n(x) - f_n(a)$. $\quad \because \lim_{n\to\infty} \int_a^x f_n'(t)dt = \lim_{n\to\infty} \{f_n(x) - f_n(a)\} = f(x) - f(a)$.

$$\because \forall x \in [a,b],\ \int_a^x g(t)dt = f(x) - f(a).$$

Since, g is continuous on [a, b].

Hence, we have $f'(x) = g(x),\ \forall x \in [a,b]$.

Theorem : If a sequence $\{f_n\}$ converges uniformly on [a, b] to a function f and if $c \in [a, b]$ such that $\lim_{x\to c} f_n(x) = a_n,\ n \in N$., then

(*i*) $\{a_n\}$ converges

(*ii*) $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to c} f_n(x)$ i.e., $\lim_{x\to c} f(x) = \lim_{n\to\infty} a_n$.

[C.H.10, 05, V.H'04]

Proof : (*i*) Let us choose $\in > 0$. Since the sequence $\{f_n\}$ is uniformly convergent, $\exists$ a natural number k such that

$$\forall x \in [a,b],\ |f_m(x) - f_n(x)| < \frac{\in}{2},\ \forall m,\ n \geq k \quad \text{---------- (1)}$$

As, $\lim_{x\to c} f_n(x) = a_n$ and $\lim_{x\to c} f_m(x) = a_m$, it follows that $\lim_{x\to c}\{f_m(x) - f_n(x)\} = a_m - a_n$.

And so $\lim_{x\to c}|f_m(x) - f_n(x)| = |a_m - a_n|$ ---------- (2)

Taking limit as $x \to c$ in (1) and using (2), we have

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$|a_m - a_n| \leq \frac{\in}{2} < \in, \ \forall m, \ n \geq k$.

This shows that $\{a_n\}$ is a Cauchy's sequence in R and therefore it is convergent.

(*ii*) Let $\lim_{n\to\infty} a_n = \ell$. Let us choose $\in > 0$.

Since, the sequence $\{f_n\}$ converges uniformly on [a, b], $\exists$ a natural number k_1 such that,

$\forall x \in [a,b], |f_n(x) - f(x)| < \frac{\in}{3}, \forall n \geq k_1$.

Since, $\lim_{n\to\infty} a_n = \ell$, $\exists$ a natural number k_2 such that $|a_n - \ell| < \frac{\in}{3}$, $\forall n \geq k_2$

Let $k = \max\{k_1, k_2\}$.

Then, $|f_k(x) - f(x)| < \frac{\in}{3}, \ \forall x \in [a,b]$ -- (3)

And $|a_k - \ell| < \frac{\in}{3}$ -- (4)

Since, $\lim_{x\to c} f_k(x) = a_k$, $\exists$ a positive δ such that

$|f_k(x) - a_k| < \frac{\in}{3}$, *whenever* $|x - c| < \delta$ -- (5)

Now, $|f(x) - \ell| \leq |f(x) - f_k(x)| + |f_k(x) - a_k| + |a_k - \ell|$

$< \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3}$, using (3), (4), (5).

$= \in$, *whenever* $|x - c| < \delta$

This proves that $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} f(x) = \ell$.

$\because \lim_{x\to c} f(x) = \lim_{n\to\infty} a_n$.

i.e., $\lim_{x\to c} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to c} f_n(x)$. **(Proved)**

Note1 : In consequences of uniform converges of the sequence $\{f_n\}$,

$\lim_{x\to c} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to c} f_n(x)$.

This indicates that the interchange of limits permissible.

Note2 : Let I be an interval and a sequence of functions $\{f_n\}$ be uniformly convergent on I to a function f. Let $c \in I$ and each f_n be continuous at c. Then f is continuous at c.

Proof : Since, f_n is continuous at c,

$\therefore \lim_{x\to c} f_n(x) = f_n(c), \forall n \in N.$ -- (1)

Since, the sequence $\{f_n\}$ converges on I to the function f, the sequence $\{f_n(c)\}$ converges to f(c).

$\therefore$ we have, $\lim_{n\to\infty} f_n(c) = f(c)$ -- (2)

and $\lim_{n\to\infty} f_n(x) = f(x), \forall x \in I$

Also, $\lim_{x\to c} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x\to c} f_n(x)$ i.e., $\lim_{x\to c} f(x) = f(c)$, by (1) and (2)

$\Rightarrow$ f is continuous at c.

Ex: If $f_n(x) = \dfrac{3}{x+n}$, $0 \le x \le 2$; state with reason's whether $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly on [0, 2]. [C.H.-2k]

Solution : Now $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{3}{x+n} = \lim_{n\to\infty} \dfrac{\frac{3}{n}}{\frac{x}{n}+1} = 0, \forall x \in [0,2]$

$\because$ The sequence $\{f_n\}$ converges point wise on [0, 2] to the function f where

$f(x) = 0, \quad 0 \le x \le 2$

Let $M_n = \sup_{x\in[0,2]} |f_n(x) - f(x)| = \sup_{x\in[0,2]} \dfrac{3}{x+n}$ -- (1)

$= \dfrac{3}{n}$, when $x = 0$.

$\because \lim_{n\to\infty} M_n = \lim_{n\to\infty} \dfrac{3}{n} = 0.$

Thus by M_n-test the given sequence converges uniformly on [0, 2].

Alternatively : $f(x) = 0, \forall x \in [0,2]$.

Now, $\forall x \in [0,2], |f_n(x) - f(x)| = \dfrac{3}{x+n} < \dfrac{3}{n}$.

Let $\epsilon > 0$ be given.

$\therefore \forall x \in [0,2], |f_n(x) - f(x)| < \epsilon$, if $n > \dfrac{3}{\epsilon}$ -- (2)

Let $k = \left[\dfrac{3}{\epsilon}\right] + 1$, then k is a natural number and (2) is true $\forall n \ge k$.

This proves that $\{f_n\}$ is uniformly convergent on [0, 2].

Ex: Let $a < b < c$ and suppose that the sequence $\{f_n(x)\}$ converges uniformly on $[a,b]$

and $[b,c]$, show that $\{f_n(x)\}$ converges uniformly on $[a,c]$.

Solution : Since $\{f_n(x)\}$ converges uniformly on $[a,b]$.

Then by Cauchy criterion , for a pre assigned $\varepsilon > 0$ there exist a natural number k_1 (depending on ε only) such that $|f_n(x) - f(x)| < \in$, $\forall n \geq k_1$ and $\forall x \in [a,b]$.

Since $\{f_n(x)\}$ converges uniformly on $[b,c]$.

Then by Cauchy criterion , for the pre assigned $\varepsilon > 0$ there exist a natural number k_2 (depending on ε only) such that $|f_n(x) - f(x)| < \in$, $\forall n \geq k_2$ and $\forall x \in [b.c]$.

Let $k = \max\{k_1, k_2\}$

Then we have , for a pre assigned $\varepsilon > 0$ there exist a natural number k (depending on ε only) such that $|f_n(x) - f(x)| < \in$, $\forall n \geq k$ and $\forall x \in [a,b]$ and

$$|f_n(x) - f(x)| < \in, \quad \forall n \geq k, \text{ and } \forall x \in [b.c].$$

Thus we can write , for a pre assigned $\varepsilon > 0$ there exist a natural number k (depending on ε only) such that $|f_n(x) - f(x)| < \in$, $\forall n \geq k$ and $\forall x \in [a,c]$

Hence by Cauchy criterion the given sequence of functions $\{f_n(x)\}$ is converges uniformly on $[a,c]$.

Ex: For each $n \in \mathbb{N}$, let $f_n(x) = \lim\limits_{m \to \infty} (\cos n!\pi x)^{2m}$.

(i) Show that $\{f_n(x)\}$ converges point wise to the function f defined by

$$f(x) = 1, x \in \mathbb{Q}$$
$$= 0, x \in \mathbb{R} - \mathbb{Q}$$

(ii) If $[a,b]$ be a closed and bounded interval , show that f_n is integrable on $[a,b]$.

Deduce that convergence of $\{f_n(x)\}$ is not uniform.

Solution :

(i) If $x \in \mathbb{R} - \mathbb{Q}$, then $0 < (\cos n!\pi x)^2 < 1$

$$\Rightarrow \lim_{m \to \infty} (\cos n!\pi x)^{2m} = 0 \Rightarrow f_n(x) = 0 \text{ for } x \in \mathbb{R} - \mathbb{Q}$$

If $x \in \mathbb{Q}$ then $x = \frac{p}{q}$ where p and q are integers and $q \geq 1$.

Thus, $(\cos n!\pi x)^{2m} = 1$ if $n \geq q \Rightarrow f_n(x) = 1$ if $n \geq q$ when $x \in \mathbb{Q}$.

Therefore,

$$\lim_{n\to\infty} f_n(x) = 1, x \in \mathbb{Q}$$
$$= 0, x \in \mathbb{R} - \mathbb{Q}$$

So $\{f_n(x)\}$ converges point wise to the function f defined by

$$f(x) = 1, x \in \mathbb{Q}$$
$$= 0, x \in \mathbb{R} - \mathbb{Q}$$

(*ii*) If $x \in [a,b] - \mathbb{Q}$, then $0 < (\cos n!\pi x)^2 < 1$

$$\Rightarrow \lim_{m\to\infty} (\cos n!\pi x)^{2m} = 0 \Rightarrow f_n(x) = 0 \text{ for } x \in \mathbb{R} - \mathbb{Q}$$

Let $x \in [a,b] \cap \mathbb{Q}$

Then for $x = \frac{p}{n!}$ where p is any integer,

$$(\cos n!\pi x)^2 = (\cos p\pi)^2 = 1$$
$$\Rightarrow f_n(x) = 1$$

For $x \neq \frac{p}{n!}$, $0 \le (\cos n!\pi x)^2 < 1$

$$\Rightarrow \lim_{m\to\infty} (\cos n!\pi x)^{2m} = 0 \Rightarrow f_n(x) = 0 \text{ for } x \neq \frac{p}{n!}$$

Thus for each $n \in N$

$$f_n(x) = 1, x \in [a,b] \cap S_n$$
$$= 0, x \in [a,b] - S_n$$

,where $S_n = \left\{0, \pm\frac{1}{n!}, \pm\frac{2}{n!}, \pm\frac{3}{n!}, \ldots\ldots\ldots\ldots\ldots\ldots\right\}$

Since each f_n is continuous on $[a,b]$ except on a countable set. Thus each f_n is integrable on $[a,b]$.

Deduce part: On $[a,b]$, $\{f_n(x)\}$ converges point wise to the function f defined by

$$f(x) = 1, x \in \mathbb{Q} \cap [a,b]$$
$$= 0, x \in [a,b] - \mathbb{Q}$$

Since f is discontinuous on $[a,b]$. Thus f is not integrable on $[a,b]$.

Hence the convergence of $\{f_n(x)\}$ is not uniform on $[a,b]$.

Example 8: Test the following sequences for uniform convergence:

(i) $\left\{ \frac{\sin nx}{\sqrt{n}} \right\}$, $0 \le x \le 2\pi$.

(ii) $\left\{ \frac{2n^2x^2}{e^{n^2x^2}} - 1 \right\}$, $\forall x \in R$

(iii) $\left\{ 1 - \frac{x^n}{n} \right\}$, $x \in [0,1]$.

Ans: (i) Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}} = 0$,

since $|\sin nx| \le 1$ i.e bounded for $\forall x \in [0, 2\pi]$,

$\therefore |f_n(x) - f(x)| = \left| \frac{\sin nx}{\sqrt{n}} \right| \le \frac{1}{\sqrt{n}}$, $\forall x \in [0, 2\pi]$,

$\therefore M_n = \sup_{x\in[0,2\pi]} |f_n(x) - f(x)| = \frac{1}{\sqrt{n}}$.

and $M_n \to 0$ as $n \to \infty$.

Therefore $\left\{ \frac{\sin nx}{\sqrt{n}} \right\}$ converges uniformly on $[0, 2\pi]$.

(ii) Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{2n^2x^2}{e^{n^2x^2}} - 1$

$= -1, \forall x \in R$.

Since $\lim_{n\to\infty} \frac{2n^2x^2}{e^{n^2x^2}} \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{4nx^2}{2nx^2e^{n^2x^2}}$

$= \lim_{n\to\infty} \frac{2}{e^{n^2x^2}} = 0$,

$$|f_n(x) - f(x)| = \left| \frac{2n^2x^2}{e^{n^2x^2}} - 1 + 1 \right| = \frac{2n^2x^2}{e^{n^2x^2}}.$$

Let $y = |f_n(x) - f(x)| = \frac{2n^2x^2}{e^{n^2x^2}}$.

$$\therefore \frac{dy}{dx} = \frac{e^{n^2x^2} \cdot 4n^2x - 2n^2x^2 \cdot e^{n^2x^2} \cdot 2n^2x}{\left(e^{n^2x^2}\right)^2}$$

$$= \frac{4n^2x - 4n^4x^3}{e^{n^2x^2}} = 0$$

$$\Rightarrow 4n^2x - 4n^4x^3 = 0 \Rightarrow 4n^2x(1 - n^2x^2) = 0,$$

$$\Rightarrow x = \pm \frac{1}{n}, \Rightarrow x \in \mathbb{R}.$$

Now
$$\frac{d^2y}{dx^2} = \frac{(4n^2 - 12n^4x^2)e^{n^2x^2} - (4n^2x - 4n^4x^3)e^{n^2x^2} \cdot 2n^2x}{\left(e^{n^2x^2}\right)^2}$$

$$= \frac{4n^2 - 12n^4x^2 - 8n^4x^2 + 8n^6x^4}{e^{n^2x^2}}$$

$$= \frac{4n^2 - 20n^4x^2 + 8n^6x^4}{e^{n^2x^2}}.$$

At $x = \pm\frac{1}{n}$, $\left.\frac{d^2y}{dx^2}\right|_{x=\pm\frac{1}{n}} = \frac{4n^2 - 20n^4 \cdot \frac{1}{n^2} + 8n^6 \cdot \frac{1}{n^4}}{e^1}$

$$= \frac{-8n^2}{e} < 0.$$

Hence y is maximum at $x = \pm\frac{1}{n}$ and, maximum value is $\frac{2}{e}$.

Hence $M_n = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \frac{2}{e}$ for $x = \frac{1}{n}$ or $x = -\frac{1}{n}$.

$\Rightarrow M_n \not\to 0$ as $n \to \infty$.

Hence $\left\{ \frac{2n^2x^2}{e^{n^2x^2}} - 1 \right\}$ is not uniformly convergent on $\mathbb{R}$.

(iii) Here $f_n(x) = 1 - \frac{x^n}{n}$, $0 \le x \le 1$.

For $x = 0$, $f(x) = \lim_{n\to\infty} f_n(x) = 1$.

For $x = 1$, $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = 1$.

For $0 < x < 1$, $f(x) = \lim_{n\to\infty} f_n(x) = 1$. as $\lim_{n\to\infty} x^n = 0$,

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Again for $a<x<b$,

$$\left|\int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt\right| = \left|\int_a^x \{f_n(t) - f(t)\}\,dt\right|$$

$$\leq \int_a^x |f_n(t) - f(t)|\,dt < \frac{x-a}{(b-a)}\cdot\epsilon < \epsilon$$

$\forall n \geq m$ and $\forall x \in [a,b]$.

$\Rightarrow \left\{\int_a^x f_n(t)\,dt\right\}$ converges uniformly to $\int_a^x f(t)\,dt$ on $[a,b]$.

Example: 10. For each $n \geq 2$. Let $f_n(x) = \begin{cases} n^2 x, & 0 \leq x \leq \frac{1}{n} \\ -n^2 x + 2n, & \frac{1}{n} < x < \frac{2}{n} \\ 0, & \frac{2}{n} \leq x \leq 1. \end{cases}$

(i) show that the sequence of functions $\{f_n\}$, $(n \geq 2)$ converges pointwise to the limit function f on $[0,1]$ where $f(x) = 0$, $0 \leq x \leq 1$.

(ii) Show that the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$.

Ans: Let $a \in (0,1)$. Then there exists a natural number $m > 2$ such that $0 < \frac{2}{m} < a$. Hence for all $n \geq m$

$0 < \frac{2}{n} < a < 1$.

$\Rightarrow f_n(a) = 0$ for all $n \geq m$ $\Rightarrow \lim_{n\to\infty} f_n(a) = 0$.

Since for $n \geq 2$, $f_n(0) = 0$, $f_n(1) = 0$: as a is any point in $(0,1)$, so $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$ for all $n \geq 2$.

Thus the sequence $\{f_n\}$, $(n \geq 2)$ converges pointwise to f on $[0,1]$ where $f(x) = 0$, $x \in [0,1]$.

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Since for all $n \geq 2$, f_n is continuous on $[0,1]$, so f_n is integrable on $[0,1]$ for all $n \geq 2$.

Now for $n \geq 2$, $\int_0^1 f_n(x)\,dx = \int_0^{1/n} f_n(x)\,dx + \int_{1/n}^{2/n} f_n(x)\,dx + \int_{2/n}^{1} f_n(x)\,dx$

$$= \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n} (-n^2 x + 2n)\,dx + \int_{2/n}^{1} 0\cdot dx$$

$$= \left[n^2 \frac{x^2}{2}\right]_0^{1/n} + \left[-n^2\frac{x^2}{2} + 2nx\right]_{1/n}^{2/n} + [0]_{2/n}^{1}$$

$$= \frac{1}{2} + \left[-2 + 4 + \frac{1}{2} - 2\right]$$

$$= 1.$$

and $\int_0^1 f(x)\,dx = \int_0^1 0\cdot dx = 0$

Now $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} [1] = 1 \neq \int_0^1 f(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$

Hence the sequence $\{f_n\}$, $n \geq 2$ is not uniformly convergent to f on $[0,1]$ where $f(x) = 0$, $\forall n \geq 2$.

<u>Example: 11.</u> Let ϕ be continuous on $[0,1]$ and $f_n(x) = x^n \phi(x)$, $x \in [0,1]$. If $\phi(1) = 0$, show that $\{f_n\}$ converges uniformly on $[0,1]$.

<u>Ans:</u> Since ϕ is continuous on $[0,1]$, $\exists$ exist a real number K such that $|\phi(x)| \leq K$ $\forall x \in [0,1]$

$\Rightarrow \phi$ is bounded on $[0,1]$.

For $0 \leq x < 1$, $\lim_{n\to\infty} x^n \phi(x) = 0$

as for $x = 1$, $\lim_{n\to\infty} x^n \phi(x) = \phi(1) = 0$ (given).

Hence $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n \phi(x) = 0$ $\forall x \in [0,1]$.

Thus the sequence $\{f_n\}$ given by $f_n(x) = x^n \phi(x)$, $x \in [0,1]$ converges pointwise to the function f on $[0,1]$ where $f(x) = 0$, $\forall x \in [0,1]$.

Let $M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$

$$= \sup\{ x^n |\phi(x)| : x \in [0,1]\}$$

Since $\phi(1) = 0$ and $x^n \phi(x) = 0$ for $x = 0$,

$\therefore$ Supremum of $\{x^n|\phi(x)| : x \in [0,1]\}$ is not attained at $x = 1$ and $x = 0$. Again $x^n |\phi(x)|$ is bounded on $[0,1]$ the supremum will be attained at some point $a \in (0,1)$.

Thus $M_n = a^n |\phi(a)| \longrightarrow 0$ as $n \to \infty$ for $0 < a < 1$.

Then by Weierstrass M-test, the sequence $\{f_n\}$ converges uniformly to f on $[0,1]$.

Example: 12 Let $f_n(x) = n^2 x (1-x^2)^n$, $x \in [0,1]$. Show that the sequence $\{f_n\}$ converges pointwise but not uniformly on $[0,1]$.

Ans: For $x = 0$, $f_n(x) = 0$ and for $x = 1$, $f_n(x) = 0$, $\forall n \in \mathbb{N}$.

$$\Rightarrow f(x) = \lim_{n \to \infty} f_n(x) = 0 \text{ for } x = 0 \text{ and } x = 1.$$

For $0 < x < 1$, $0 < x^2 < 1$. $\Rightarrow 0 < 1 - x^2 < 1$.

For every fixed $x \in (0,1)$.

$$\lim_{n \to \infty} \frac{f_{n+1}(x)}{f_n(x)} = \lim_{n \to \infty} \frac{(n+1)^2 x (1-x^2)^{n+1}}{n^2 x (1-x^2)^n}$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^2 \cdot (1 - x^2)$$

$$= (1 - x^2) < 1.$$

Hence $\lim_{n \to \infty} f_n(x) = 0$, for $0 < x < 1$.

$\therefore f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x=0, \text{ or } x=1, \\ 0 & 0<x<1, \end{cases}$

$= 0, \ \forall x \in [0,1]$.

Thus $\{f_n\}$ converges pointwise to f on $[0,1]$ where $f(x)=0, x\in[0,1]$.

Now $\int_0^1 f_n(x)\,dx = n^2 \int_0^1 x(1-x^2)^n\,dx = \frac{n^2}{2}\left[-\frac{(1-x^2)^{n+1}}{n+1}\right]_0^1$

$= \frac{n^2}{2(n+1)}$

$\therefore \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = +\infty$, whereas $\int_0^1 f(x)\,dx = 0$,

Thus $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx$. which by sufficient condition of uniform convergence, implies $\{f_n\}$ is not uniformly convergent on $[0,1]$.

Uniform Convergence, Limit and Limit Point:

Theorem: Let D be a subset of $\mathbb{R}$ and a sequence of functions $\{f_n\}$ be uniformly convergent on D to a function f. Let $x_0 \in D'$ [the derived set of D], i.e x_0 is the limit point or accumulation point of D. and $\lim_{x\to x_0} f_n(x) = a_n$.

Then (i) the sequence $\{a_n\}$ is convergent

(ii) $\lim_{x\to x_0} f(x)$ exists.

and (iii) $\lim_{x\to x_0} f(x) = \lim_{n\to\infty} a_n$ i.e $\lim_{n\to\infty}\lim_{x\to x_0} f_n(x) = \lim_{x\to x_0}\lim_{n\to\infty} f_n$.

Proof: (i) Let us choose $\epsilon>0$. Since the sequence $\{f_n\}$ is uniformly convergent, there exists a natural number k such that (by Cauchy Criterion)

$\forall x \in D$. $|f_m(x) - f_n(x)| < \epsilon/2$ for all $m, n \geq k$ —— (1)

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