Semester-II

Course Type- Core-3

Course Title- C3T: Real Analysis

Topic- Sub sequence

References- S.K. Mapa Book.

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Real Sequence 2

Subsequence: Let $\{u_n\}$ be a real sequence and $\{r_n\}$ be strictly increasing sequence of natural numbers, i.e., $r_1 < r_2 < r_3 < \dots\dots\dots < r_n < \dots\dots\dots$. Then the sequence $\{u_{r_n}\}$ is said to be a subsequence of the sequence $\{u_n\}$. The elements of the subsequence $\{u_{r_n}\}$ are $u_{r_1}, u_{r_2}, \dots\dots\dots, u_{r_n}, \dots\dots\dots$

Example: Let $u_n = \frac{1}{n}$ and $r_n = 2n$ for all $n \in \mathbb{N}$.

Then $\{u_{r_n}\} = \{u_2, u_4, u_6, \dots\dots\dots\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\dots\dots\right\}$ is a subsequence of $\left\{\frac{1}{n}\right\}$.

Theorem: If a sequence $\{u_n\}$ converges to l then every subsequence of $\{u_n\}$ also converges to l. **VU'1997' 00**

Proof: Let $\{r_n\}$ be strictly increasing sequence of natural numbers. Then $\{u_{r_n}\}$ is a subsequence of the sequence $\{u_n\}$.

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} u_n = l$, there exists a natural number k such that $l - \varepsilon < u_n < l + \varepsilon$ for all $n \geq k$

Since $\{r_n\}$ is a strictly increasing sequence of natural numbers, there exists a natural number k_0 such that $r_n > k$ for all $n \geq k_0$.

Therefore $l - \varepsilon < u_{r_n} < l + \varepsilon$ for all $n \geq k_0$

i.e., $|u_{r_n} - l| < \varepsilon$ for all $n \geq k_0$

Since ε is arbitrary, $\lim_{n\to\infty} u_{r_n} = l$

Ex 1: Prove that $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = \sqrt{e}$. **VU'2001, 07**

✍ Let $u_n = \left(1+\frac{1}{n}\right)^n$, $v_n = \left(1+\frac{1}{2n}\right)^{2n}$ and $w_n = \left(1+\frac{1}{2n}\right)^n$ for all $n \in \mathbb{N}$.

$\{u_n\}$ is a convergent sequence and $\lim_{n\to\infty} u_n = e$.

Since $v_n = u_{2n}$ for all $n \in \mathbb{N}$, $\{v_n\}$ is a subsequence of $\{u_n\}$ and therefore $\lim_{n\to\infty} v_n = e$.

Now $w_n = \sqrt{v_n}$ for all $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} w_n = \lim_{n\to\infty}\sqrt{v_n} = \sqrt{e}$

Ex 2: Prove that the sequence $\{(-1)^n\}$ is not convergent. **VU'2000**

✍ Let $u_n = (-1)^n$, $v_n = u_{2n}$, $w_n = u_{2n-1}$.

Then $\{v_n\}$ is the subsequence $\{1,1,1,\dots\dots\dots\}$ and $\lim_{n\to\infty} v_n = 1$, $\{w_n\}$ is the subsequence $\{-1,-1,-1,\dots\dots\dots\}$ and $\lim_{n\to\infty} w_n = -1$.

Since two different subsequences converge to two different limits, the sequence $\{u_n\}$ is not convergent.

Theorem: If the subsequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ of a sequence $\{u_n\}$ converge to the same limit l then the sequence $\{u_n\}$ is convergent and $\lim_{n\to\infty} u_n = l$.

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Note1: If two subsequences of a sequence converge to the same limit l, the sequence $\{u_n\}$ may not be convergent.

For example, let $u_n = \sin\frac{n\pi}{4}$.

Then the subsequence $\{u_{8n-7}\}$ is $\left\{\sin\frac{\pi}{4}, \sin\frac{9\pi}{4}, \sin\frac{17\pi}{4}, \ldots\ldots\ldots\right\}$ and this converges to $\frac{1}{\sqrt{2}}$.

The subsequence $\{u_{8n-5}\}$ is $\left\{\sin\frac{3\pi}{4}, \sin\frac{11\pi}{4}, \sin\frac{19\pi}{4}, \ldots\ldots\ldots\right\}$ and this converges to $\frac{1}{\sqrt{2}}$.

But the sequence $\{u_n\}$ is not convergent.

Note2: If $k \in \mathbb{N}$ and k subsequences $\{u_{kn}\}$, $\{u_{kn-1}\}$, $\{u_{kn-2}\}$, ………, $\{u_{kn-k+1}\}$ converges to the same limit l then the sequence $\{u_n\}$ is convergent and $\lim_{n\to\infty} u_n = l$.

Theorem: Every subsequence of a monotone increasing (decreasing) sequence of real numbers is monotone increasing (decreasing).

Theorem: A monotone sequence of real numbers having a convergent subsequence with limit l, is convergent with limit l.

Theorem: A monotone sequence of real numbers having a divergent subsequence is properly divergent.

Ex 3: Let $\{u_n\}$ be a sequence defined by $0 < u_1 < u_2$ and $u_{n+2} = \frac{1}{2}(u_n + u_{n+1})$. Prove that both the sub-sequence $\{u_{2n}\}$ and $\{u_{2n-1}\}$ converge to the same limit.

$u_3 - u_1 = \frac{u_1 + u_2}{2} - u_1 = \frac{u_2 - u_1}{2} > 0$ i.e. $u_1 < u_3$

$u_3 - u_2 = \frac{u_1 + u_2}{2} - u_2 = \frac{u_1 - u_2}{2} < 0$ i.e. $u_3 < u_2$ and $u_4 - u_2 = \,,\ u_4 - u_3 = ?$

So, $u_1 < u_3 < u_2$. Similarly $u_3 < u_4 < u_2, u_3 < u_5 < u_4, u_5 < u_6 < u_4$………

This inequality gives $u_1 < u_3 < u_5 < \ldots\ldots\ldots < u_6 < u_4 < u_2$

This shows that the sequence $\{u_{2n-1}\}$ is a monotone increasing sequence bounded above, u_2 is an upper bound

Also the sequence $\{u_{2n}\}$ is a monotone decreasing sequence bounded below, u_1 being a lower bound.

Thus both the sequences $\{u_{2n-1}\}$ and $\{u_{2n}\}$ are convergent Let $\lim_{n\to\infty} u_{2n} = l$ and $\lim_{n\to\infty} u_{2n-1} = m$. Now from the given relation $2u_{2n+2} = u_{2n} + u_{2n+1}$

$$\lim_{n\to\infty} 2u_{2n+2} = \lim_{n\to\infty} u_{2n} + \lim_{n\to\infty} u_{2n+1}$$

$$\Rightarrow 2l = l + m \Rightarrow l = m$$

Ex 4: A sequence $\{u_n\}$ defined by $u_n > 0$ and $u_{n+1} = \frac{6}{1+u_n}$ for all $n \in \mathbb{N}$

(i)Prove that the sub-sequences $\{u_{2n-1}\}$ and $\{u_{2n}\}$ converges to common limit

(ii)Find $\lim_{n\to\infty} u_n$

$$u_{n+1}-u_n=\frac{6}{1+u_n}-u_n=\frac{6-u_n-u_n^2}{1+u_n}=\frac{(2-u_n)(3+u_n)}{1+u_n}$$

Therefore $u_n<2\Rightarrow u_{n+1}=\frac{6}{1+u_n}>2;$

$u_n>2\Rightarrow u_{n+1}=\frac{6}{1+u_n}<2$ Combining the two cases we get

$u_n<2\Rightarrow u_n<2<u_{n+1};u_n>2\Rightarrow u_{n+1}<2<u_n$ .... (i)

$$u_{n+2}-u_n=\frac{6(1+u_n)}{7+u_n}-u_n=\frac{6-u_n-u_n^2}{7+u_n}=\frac{(2-u_n)(3+u_n)}{7+u_n}$$

$u_n<2\Rightarrow u_n<u_{n+2};u_n>2\Rightarrow u_n>u_{n+2}$..........................(ii)

Case 1: Let $u_1<2$. Then $u_2>2$.

From (i) $u_1<2\Rightarrow u_1<2<u_2;u_2>2\Rightarrow u_3<2<u_2;u_3<2\Rightarrow u_3<2<u_4;u_4>2\Rightarrow u_5<2<u_4;$.....

From (ii) $u_1<2\Rightarrow u_1<u_3;\ u_3<2\Rightarrow u_3<u_5;$ ....

$u_2>2\Rightarrow u_2>u_4;\ u_4>2\Rightarrow u_4>u_6;$...............

Therefore $u_1<u_3<u_5<$..............$<u_6<u_4<u_2$

This shows that the sequence $\{u_{2n-1}\}$ is a monotone increasing sequence, bounded above and the sequence $\{u_{2n}\}$ is a monotone decreasing sequence, bounded below. Hence both the subsequences are convergent.

Let $\lim_{n\to\infty}u_{2n-1}=l,\lim_{n\to\infty}u_{2n}=m$. From the relation we have $u_{2n}=\frac{6}{1+u_{2n-1}},u_{2n+1}=\frac{6}{1+u_{2n}}$ for all $n\in\mathbb{N}$

Taking limit as $n\to\infty$, we have $m=\frac{6}{1+l},l=\frac{6}{1+m}$. Therefore $l=m$ and the sub-sequences $\{u_{2n-1}\}$ and $\{u_{2n}\}$ converges to a common limit.

Case 2: $u_1>2$

From (i) and (ii) we get $u_2<u_4<u_6<$........$<u_5<u_3<u_1$.

This shows that the sequence $\{u_{2n-1}\}$ is a monotone decreasing sequence, bounded above and the sequence $\{u_{2n}\}$ is a monotone increasing sequence, bounded below. Hence both the subsequences are convergent.

Let $\lim_{n\to\infty}u_{2n-1}=l,\lim_{n\to\infty}u_{2n}=m$. From the relation we have $u_{2n}=\frac{6}{1+u_{2n-1}},u_{2n+1}=\frac{6}{1+u_{2n}}$ for all $n\in\mathbb{N}$

Taking limit as $n\to\infty$, we have $m=\frac{6}{1+l},l=\frac{6}{1+m}$. Therefore $l=m$ and the sub-sequences $\{u_{2n-1}\}$ and $\{u_{2n}\}$ converges to a common limit.

(ii) Let the limit be l. We have $u_{n+1}=\frac{6}{1+u_n}$ for all $n\in\mathbb{N}$. Taking limit as $n\to\infty$, we have $l=\frac{6}{1+l}$. This gives $l=2$ or $l=-3$.

Since $\{u_n\}$ is a sequence of +ve real numbers therefore $l\neq-3$. Therefore $l=2$

Ex 5: A sequence $\{x_n\}$ is defined as follows $x_2\le x_4\le x_6\le.....\le x_5\le x_3\le x_1$ and $\{y_n\}$ be defined by $y_n=x_{2n-1}-x_{2n}$ such that $y_n\to0$ as $n\to\infty$. Show that $\{x_n\}$ is convergent.

CU'2005

Clearly $\{x_{2n}\}$ is a monotone increasing sequence bounded above (x_1 is an upper bound). Also the sequence $\{x_{2n-1}\}$ is a monotone decreasing sequence bounded below (x_2 is a lower bound). Thus the two subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ are convergent

Again since $y_n \to 0$ as $n \Rightarrow \infty \Rightarrow \lim\limits_{n\to\infty} x_{2n} = \lim\limits_{n\to\infty} x_{2n-1}$

Thus the two subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ are convergent to the same limit

$\Rightarrow \{x_n\}$ is convergent

Ex 6: $\{x_n\}$ and $\{y_n\}$ are bounded sequences and a sequence $\{z_n\}$ is defined by $z_1 = x_1, z_2 = y_2, z_3 = x_2, z_4 = y_2, z_5 = x_3, z_6 = y_3, ...$ Prove that the sequence $\{z_n\}$ is convergent iff both the sequences $\{x_n\}$ and $\{y_n\}$ are convergent with the same limit.

Clearly $\{z_{2n-1}\} = \{x_n\}$ and $\{z_{2n}\} = \{y_n\}$

Let $\{z_n\}$ is convergent. Then both the sub-sequences $\{z_{2n-1}\}$ and $\{z_{2n}\}$ of the sequence $\{z_n\}$ are convergent with the same limit

$\Rightarrow$ Both the sequences $\{x_n\}$ and $\{y_n\}$ are convergent with the same limit

Conversely, let both the sequences $\{x_n\}$ and $\{y_n\}$ are convergent with the same limit

$\Rightarrow$ Both the sub-sequences $\{z_{2n-1}\}$ and $\{z_{2n}\}$ of the sequence $\{z_n\}$ are convergent with the same limit

$\Rightarrow$ The sequence $\{z_n\}$ is convergent

Thus the sequence $\{z_n\}$ is convergent iff both the sequences $\{x_n\}$ and $\{y_n\}$ are convergent with the same limit

Theorem: Every sequence of real numbers has a monotone subsequence.

Proof: Let $\{u_n\}$ be a sequence of real numbers. An element u_k is said to be a peak of the sequence $\{u_n\}$ if $u_k \ge u_n$ for all $n > k$, i.e., u_k is greater than or equal to all subsequent elements beyond u_k. A sequence may or may not have a peak or else it may have a finite or an infinite number of peaks.

We consider the following cases.

Case1: Let the sequence $\{u_n\}$ have infinitely many peaks.

Let the peaks be $u_{r_1}, u_{r_2}, ...$ (u_{r_1} being the first peak, u_{r_2} be the second peak,..............).

Then $u_{r_1} \ge u_{r_2} \ge u_{r_3}...............$

The subsequence $\{u_{r_1}, u_{r_2}, u_{r_3}, ...\}$ is a monotone decreasing sequence.

Case2: Let the sequence have either no peak or a finite number of peaks.

Let the peaks be arranged in ascending order of the subscripts as $u_{r_1}, u_{r_2}, ..., u_{r_m}$. Let $s_1 = r_m + 1$.

Then u_{s_1} is not a peak and there is no peak beyond the element u_{s_1}.

Since u_{s_1} is not a peak, there is an $s_2 \in \mathbb{N}$ with $s_2 > s_1$ such that $u_{s_2} > u_{s_1}$.

Since u_{s_2} is not a peak, there is an $s_3 \in \mathbb{N}$ with $s_3 > s_2$ such that $u_{s_3} > u_{s_2}$.

Proceeding thus we obtain natural numbers s_i such that $s_1 < s_2 < s_3 < ....$

and $u_{s_1} < u_{s_2} < u_{s_3} < ....$

Clearly, the subsequence $\{u_{s_n}\}$ is a monotone increasing sequence of the sequence $\{u_n\}$.
This completes the proof.

Sub sequential limit: Let $\{u_n\}$ be a real sequence. A real number l is said to be a sub sequential limit of the sequence $\{u_n\}$ if there exists a subsequence of $\{u_n\}$ that converges to l.

Theorem: A real number l is a sub sequential limit of a sequence $\{u_n\}$ if and only if every neighbourhood of l contains infinitely many elements of the sequence $\{u_n\}$.

Bolzano-Weierstrass theorem: Every bounded sequence of real numbers has a convergent subsequence. **VU'2002, 06, CU'2001, 03**

Proof: Let $\{u_n\}$ be a bounded sequence. Then there is a closed and bounded interval, say $I=[a,b]$, such that $u_n \in I$ for every $n\in\mathbb{N}$.

Let $c=\dfrac{a+b}{2}$ and $I'=[a,c]$, $I''=[c,b]$. Then at least one of the intervals I' and I'' contains infinitely many elements of $\{u_n\}$.

Let $I_1=[a_1,b_1]$ be one such interval. Then $I_1\subset I$ and $|I_1|$=the length of the interval$=\frac{1}{2}(b-a)$.

Let $c_1=\dfrac{a_1+b_1}{2}$ and $I_1'=[a_1,c_1]$, $I_1''=[c_1,b_1]$. Then at least of the intervals contains infinitely many elements of $\{u_n\}$. Let $I_2=[a_2,b_2]$ be such an interval.

Then $I_2\subset I_1$ and $|I_2|=\frac{1}{2}|I_1|$.

Continuing thus, we obtain a sequence of closed and bounded intervals $\{I_n\}$ such that

(i) $I_{n+1}\subset I_n$ for all $n\in\mathbb{N}$.

(ii) $|I_n|=\dfrac{1}{2^n}(b-a)$ and therefore $\lim\limits_{n\to\infty}|I_n|=0$; and

(iii) each I_n contains infinitely many elements of $\{u_n\}$.

By Cantor's theorem on nested intervals, there exists a unique point α such that $\alpha\in\bigcap\limits_{n=1}^{\infty} I_n$.

We prove that α is a sub sequential limit of the sequence $\{u_n\}$.

Let us choose $\varepsilon>0$. There exists a natural number k such that $0<\dfrac{b-a}{2^k}<\varepsilon$. That is, $|I_k|<\varepsilon$

Since $\alpha\in I_k$ and $|I_k|<\varepsilon$, I_k is contained in the neighbourhood $(\alpha-\varepsilon,\alpha+\varepsilon)$ and consequently, the ε-neighbourhood of α contains infinitely many elements of $\{u_n\}$.

Since ε is arbitrary, each neighbourhood of α contains infinitely many elements of $\{u_n\}$. Therefore α is a sub sequential limit of $\{u_n\}$.

Therefore there exists a subsequence of $\{u_n\}$ that converges to α. In other words, $\{u_n\}$ has a convergent subsequence.

Definition: Let $\{u_n\}$ be a bounded sequence of real numbers. The greatest sub sequential limit of $\{u_n\}$ is said to be the **upper limit** or the **limit superior** of $\{u_n\}$ and this is denoted by $\overline{\lim}\, u_n$ or

$\limsup\limits_{n\to\infty} u_n$. The least sub sequential limit of $\{u_n\}$ is said to be the **lower limit** or the **limit inferior** of $\{u_n\}$ and this is denoted by $\underline{\lim}\, u_n$ or $\liminf\limits_{n\to\infty} u_n$.

If $\{u_n\}$ is unbounded above then we define $\overline{\lim}\, u_n = \infty$.

If $\{u_n\}$ is unbounded below then we define $\underline{\lim}\, u_n = -\infty$.

Examples:

1. Let $u_n = (-1)^n\left(1+\dfrac{1}{n}\right), n \geq 1$. Then the sequence $\{u_n\}$ is bounded sequence. $\overline{\lim}\, u_n = 1$, $\underline{\lim}\, u_n = -1$.

2: Let $u_n = \dfrac{1}{n}, n \geq 1$. Then the sequence $\{u_n\}$ is bounded sequence. $\overline{\lim}\, u_n = \underline{\lim}\, u_n = 0$.

3: Let $u_n = (-1)^n n^2, n \geq 1$. Then the sequence $\{u_n\}$ is unbounded above and unbounded below. $\overline{\lim}\, u_n = \infty$, $\underline{\lim}\, u_n = -\infty$.

4: Let $u_n = n^{(-1)^n \cdot n}, n \geq 1$. Then the sequence $\{u_n\}$ is unbounded above and bounded below. $\overline{\lim}\, u_n = \infty$, $\underline{\lim}\, u_n = 0$.

Ex 7: Find the upper and lower limit of the sequence $\{a_n\}$ where $a_n = (-1)^n\left(1+\dfrac{1}{n}\right)(n = 1, 2, 3, ...)$. Find a sub-sequence of this sequence that converges to the lower limit. **CU'2000**

(H.W)

Ex 8: Find the upper and lower limit of the sequence $\{a_n\}$ where $a_n = \left(1-\dfrac{1}{n^2}\right)\sin\dfrac{n\pi}{2}$. Find a sub-sequence of this sequence that converges to the lower limit.

VU'2004, CU'1999, 01, 07

(H.W.)

Properties of Upper limit and Lower limit:

Let $\{u_n\}$ be a bounded sequence and $u^* = \overline{\lim}\, u_n$, $u_* = \underline{\lim}\, u_n$.

The upper limit u^* satisfies the following conditions:

For each positive ε,

(i) $u_n > u^* - \varepsilon$ For infinitely many values of n, and

(ii) There exists a natural number k such that $u_n < u^* + \varepsilon$ for all $n \geq k$.

The lower limit u_* satisfies the following conditions:

For each positive ε,

(i) $u_n < u_* + \varepsilon$ For infinitely many values of n, and

(ii) There exists a natural number k such that $u_n > u_* - \varepsilon$ for all $n \geq k$.

Theorem: Let $\{u_n\}$ is a sequence of real numbers. Then $\underline{\lim}\, u_n \leq \overline{\lim}\, u_n$ **CU'2002**

Let $\underline{\lim}\, u_n = u_*, \overline{\lim}\, u_n = u^*$

If possible let the statement be not true i.e. $u_* > u^*$. Then $u_* - u^* > 0$

Choose $\varepsilon = \dfrac{1}{2}(u_* - u^*) \Rightarrow u^* + \varepsilon = u_* - \varepsilon$