Semester - II

Course Type - Core - 3

Course Title - C3T: Real Analysis

Topic: Sub sequence.

References: S. K. Mapa Book.

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— Shambhu Nath Acharya

By the definition of u_* there exists only finite number of terms of $\{u_n\}$ less than $u_* - \varepsilon$

Again by the definition of u^* there are infinite number of terms of $\{u_n\} < u^* + \varepsilon (= u_* - \varepsilon)$

The two statements are clearly contradicting each other. Therefore $u_* > u^*$ does not hold therefore $u_* \le u^* \Rightarrow \underline{\lim} u_n \le \overline{\lim} u_n$

Theorem: A bounded sequence $\{u_n\}$ is convergent if and only if $\overline{\lim} u_n = \underline{\lim} u_n$

VU'2004, CU'2005, 07

Proof: Let $\{u_n\}$ be a convergent sequence and $\lim\limits_{n\to\infty} u_n = l$.

Since $\{u_n\}$ is convergent, every subsequence of $\{u_n\}$ converges to l. Therefore l is the greatest as well as the least subsequential limit. That is, $\overline{\lim} u_n = \underline{\lim} u_n$.

Conversely, let $\{u_n\}$ be a bounded sequence such that $\overline{\lim} u_n = \underline{\lim} u_n$.

Let $\overline{\lim} u_n = \underline{\lim} u_n = l$.

Let us choose $\varepsilon > 0$.

Since $\overline{\lim} u_n = l$, there exists a natural number k_1 such that

$u_n < l + \varepsilon$ for all $n \ge k_1$.

Since $\underline{\lim} u_n = l$, there exists a natural number k_2 such that

$u_n > l - \varepsilon$ for all $n \ge k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $l - \varepsilon < u_n < l + \varepsilon$ for all $n \ge k$ i.e., $|u_n - l| < \varepsilon$ for all $n \ge k$

This proves that $\lim\limits_{n\to\infty} u_n = l$.

In other words, the sequence $\{u_n\}$ is convergent.

Theorem: Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences. Then

(i) $\overline{\lim} u_n + \overline{\lim} v_n \ge \overline{\lim}(u_n + v_n)$ **VU'2002**

(ii) $\underline{\lim} u_n + \underline{\lim} v_n \le \underline{\lim}(u_n + v_n)$. **CU'1998**

Proof: (i) Since $\{u_n\}$ and $\{v_n\}$ are both bounded sequences, the sequence $\{u_n + v_n\}$ is a bounded sequence.

Let $\overline{\lim} u_n = l_1$, $\overline{\lim} v_n = l_2$, $\overline{\lim}(u_n + v_n) = p$.

Let us choose $\varepsilon > 0$.

Since $\overline{\lim} u_n = l_1$, there exists a natural number k_1

Such that $u_n < l_1 + \frac{\varepsilon}{2}$ for all $n \ge k_1$.

Since $\overline{\lim} v_n = l_2$, there exists a natural numbers k_2.

Such that $v_n < l_2 + \frac{\varepsilon}{2}$ for all $n \ge k_2$.

Let $k = \max\{k_1, k_2\}$.

Then $u_n < l_1 + \frac{\varepsilon}{2}$ and $v_n < l_2 + \frac{\varepsilon}{2}$ for all $n \ge k$

So $u_n + v_n < l_1 + l_2 + \varepsilon$ for all $n \geq k$

It follows that no sub sequential limit of $\{u_n + v_n\}$ can be greater than $l_1 + l_2 + \varepsilon$. Since $\varepsilon (> 0)$ is arbitrary, every sub sequential limit $\leq l_1 + l_2$.

Hence $p \leq l_1 + l_2$

(ii) Similar proof.

Note (VU'2002): Strict inequality may occur. For example; if $u_n = \sin\frac{n\pi}{2}, n \in \mathbb{N}$; $v_n = \cos\frac{n\pi}{2}$, $n \in \mathbb{N}$

then

$\underline{\lim}(u_n + v_n) = -1$, $\underline{\lim} u_n = -1$, $\underline{\lim} v_n = -1$.

$\overline{\lim}(u_n + v_n) = 1$, $\overline{\lim} u_n = 1$, $\overline{\lim} v_n = 1$.

So in this case $\underline{\lim} u_n + \underline{\lim} v_n < \underline{\lim}(u_n + v_n)$ and $\overline{\lim} u_n + \overline{\lim} v_n > \overline{\lim}(u_n + v_n)$.

Theorem(Cauchy's general principle of convergence): A necessary and sufficient condition for the convergence of a sequence $\{u_n\}$ is that for a pre-assigned positive ε there exists a natural number k such that $|u_{n+p} - u_n| < \varepsilon$ for all $n \geq k$ and for $p = 1,2,3,\ldots\ldots\ldots$

Proof: Let $\{u_n\}$ be convergent and $\lim_{n\to\infty} u_n = l$.

Let $\varepsilon > 0$

Then there exists a natural number k such that $|u_n - l| < \frac{\varepsilon}{2}$ for all $n \geq k$.

Therefore $|u_{n+p} - l| < \frac{\varepsilon}{2}$ for all $n \geq k$ and $p = 1,2,3,\ldots\ldots\ldots$

Now $|u_{n+p} - u_n| \leq |u_{n+p} - l| + |u_n - l|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $n \geq k$ and $p = 1,2,3,\ldots\ldots\ldots$

This proves that the condition is necessary.

We now prove that the sequence $\{u_n\}$ is convergent under the stated condition. First we prove that the sequence $\{u_n\}$ is bounded.

Let $\varepsilon = 1$. Then there exists a natural number k such that $|u_{n+p} - u_n| < 1$ for all $n \geq k$ and $p = 1,2,3,\ldots\ldots\ldots$

Therefore $|u_{k+p} - u_k| < 1$ for $p = 1,2,3,\ldots\ldots$

Or, $u_k - 1 < u_{k+p} < u_k + 1$ for $p = 1,2,3,\ldots\ldots$

Let $B = \max\{u_1, u_2, \ldots\ldots\ldots, u_k, u_k + 1\}$ and $b = \min\{u_1, u_2, \ldots\ldots\ldots, u_k, u_k - 1\}$.

Then $b \leq u_n \leq B$ for all $n \in \mathbb{N}$

This proves that $\{u_n\}$ is a bounded sequence.

By Bolzano-Weierstrass theorem, the sequence $\{u_n\}$ has a convergent subsequence. Let l be a limit of that subsequence. Then l is a sub sequential limit of $\{u_n\}$.

Let $\varepsilon > 0$. Then by the given condition, there exists a natural number m such that $|u_{n+p} - u_n| < \frac{\varepsilon}{3}$ for all $n \geq m$ and $p = 1,2,3,\ldots\ldots\ldots$

Taking $m=n$, it follows that $|u_{m+p}-u_m|<\frac{\varepsilon}{3}$ for $p=1,2,3,........(1)$

Since l is a sub sequential limit of $\{u_n\}$, each ε-neighbourhood of l contains infinite number of elements of $\{u_n\}$. Therefore there exists a natural number $q>m$ such that $|u_q-l|<\frac{\varepsilon}{3}$.

As $q>m$, it follows from (1) that $|u_q-u_m|<\frac{\varepsilon}{3}$.

Now $|u_{q+m}-l|\le|u_{m+p}-u_m|+|u_m-u_q|+|u_q-l|$

$$<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon \text{ for } p=1,2,3,........$$

Therefore $|u_n-l|<\varepsilon$ for all $n\ge m+1$

Since ε is arbitrary, the sequence $\{u_n\}$ converges to l.

In other words, $\{u_n\}$ is a convergent sequence. This completes the proof.

Ex 9: Use Cauchy's general principle of convergence to prove that the sequence $\left\{\frac{n}{n+1}\right\}$ is convergent. **CU'2001**

Let $u_n=\frac{n}{n+1}$. Let p be a natural number.

Then $u_{n+p}=\frac{n+p}{n+p+1}$

$$|u_{n+p}-u_n|=\left|\frac{n+p}{n+p+1}-\frac{n}{n+1}\right|$$

$$=\frac{p}{(n+p+1)(n+1)}$$

$$<\frac{1}{n+1}<\frac{1}{n} \text{ for all } p\text{, since } \frac{p}{n+p+1}<1 \text{ for all } p.$$

Let $\varepsilon>0$. Then $\frac{1}{n}<\varepsilon$ holds for $n>\frac{1}{\varepsilon}$

Let $m=\left[\frac{1}{\varepsilon}\right]+1$. Then m is a natural number and $|u_{n+p}-u_n|<\varepsilon$ for all $n\ge m$ and $p=1,2,3,.........$

This proves that the sequence $\{u_n\}$ is convergent.

Ex 10: Use Cauchy's general principle of convergence to prove that the sequence $\{u_n\}$ where $u_n=1+\frac{1}{2}+\frac{1}{3}+.........+\frac{1}{n}$, is not convergent. **CU'2004**

Let p be a natural number.

$$|u_{n+p}-u_n|=\frac{1}{n+1}+\frac{1}{n+2}+..........+\frac{1}{n+p}.$$

Let us choose $n=m$ and $p=m$.

Then $|u_{2m}-u_m|=\frac{1}{m+1}+\frac{1}{m+2}+..........+\frac{1}{2m}$

$$> \frac{1}{2m} + \frac{1}{2m} + \dots\dots\dots + \frac{1}{2m} = \frac{1}{2}.$$

If we choose $\varepsilon = \frac{1}{2}$ then no natural number k can be found such that $|u_{n+p} - u_n| < \varepsilon$ will hold for all $n \geq k$ and for every natural number p.

This shows that Cauchy condition is not satisfied by the sequence and the sequence $\{u_n\}$ is not convergent.

Cauchy sequence(CU'1997, 02, 07): A sequence $\{u_n\}$ is said to be a Cauchy sequence if for a pre-assigned positive ε there exists a natural number k such that $|u_m - u_n| < \varepsilon$ for all $m, n \geq k$.

Replacing m by $n+p$ where $p = 1, 2, 3, \dots\dots\dots$ the above condition can be equivalently stated as $|u_{n+p} - u_n| < \varepsilon$ for all $n \geq k$ and $p = 1, 2, 3, \dots\dots\dots$

Theorem: A Cauchy sequence of real numbers is convergent. **VU'2004**

Proof: Let $\{u_n\}$ be a Cauchy sequence. First we prove that the sequence $\{u_n\}$ is bounded.

Let $\varepsilon = 1$. Then there exists a natural number k such that $|u_m - u_n| < 1$ for all $m, n \geq k$.

Therefore $|u_k - u_n| < 1$ for all $n \geq k$

Or, $u_k - 1 < u_n < u_k + 1$ for all $n \geq k$

Let $B = \max\{u_1, u_2, \dots\dots\dots, u_k - 1, u_k + 1\}$,

$b = \min\{u_1, u_2, \dots\dots\dots, u_k - 1, u_k + 1\}$.

Then $b \leq u_n \leq B$ for all $n \in \mathbb{N}$ and this proves that the sequence $\{u_n\}$ is bounded. By Bolzano-Weierstrass theorem, $\{u_n\}$ has a convergent subsequence.

Let l be the limit of that convergent subsequence. Then l is a sub sequential limit of $\{u_n\}$.

We now prove that the sequence $\{u_n\}$ converges to l.

Let us choose $\varepsilon > 0$. There exists a natural number k such that $|u_m - u_n| < \frac{\varepsilon}{2}$ for all $m, n \geq k$ .... (1)

Since l is a sub sequential limit of $\{u_n\}$, there exists a natural number $q > k$ such that $|u_q - l| < \frac{\varepsilon}{2}$.

Since $q > k$, from (1) $|u_q - u_n| < \frac{\varepsilon}{2}$ for all $n \geq k$.

Now $|u_n - l| \leq |u_n - u_q| + |u_q - l|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for all $n \geq k$.

That is, $|u_n - l| < \varepsilon$ for all $n \geq k$.

This implies $\lim_{n\to\infty} u_n = l$. In other words, the sequence $\{u_n\}$ is convergent.

Ex 11: Show that every Cauchy sequence is bounded **CU'2006**

Hints: Let $\{u_n\}$ is a Cauchy sequence. Then $\{u_n\}$ is convergent

Then prove the theorem that every convergent sequence is bounded

Theorem: A convergent sequence is a Cauchy sequence.

Proof: Let $\{u_n\}$ be a convergent sequence and let $\lim_{n\to\infty} u_n = l$.

Let us choose $\varepsilon > 0$. Then there exists a natural number k such that $|u_n - l| < \frac{\varepsilon}{2}$ for all $n \geq k$.

If m, n be natural numbers $\geq k$, then $|u_m - l| < \frac{\varepsilon}{2}$ and $|u_n - l| < \frac{\varepsilon}{2}$.

Now $|u_m - u_n| \leq |u_m - l| + |u_n - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ for all $m, n \geq k$

That is, $|u_m - u_n| < \varepsilon$ for all $m, n \geq k$.

This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

Ex 12: Prove that the sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence.

Let $u_n = \frac{1}{n}$. Let us choose a positive ε. By Archimedean property there exists a positive number $k\varepsilon > 2$.

Then $|u_m - u_n| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ if $m, n \geq k$

This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

Ex 13: Prove that the sequence $\left\{\frac{1}{n+1}\right\}$ is a Cauchy sequence. **CU'1997**

(H.W.)

Ex 14: Prove that the sequence $\{u_n\}$ where $u_n = \frac{n+1}{n}, n \in \mathbb{N}$ is a Cauchy sequence.

CU'2008

(H.W.)

Ex 15: Prove that $\{2^n\}$ is not a Cauchy sequence. **VU'2002, CU'2007**

Let $u_n = 2^n$ for $n \in \mathbb{N}$

Then $|u_{n+p} - u_n| = |2^{n+p} - 2^n| = 2^n |2^p - 1| \geq 2^n \geq 2$

Let $\varepsilon = 1$. Then for this chosen ε there does not exists a natural number k such that $|u_{n+p} - u_n| < \varepsilon$ for all $n \geq k$

$\Rightarrow \{2^n\}$ is not a Cauchy sequence

Ex 16: Prove that the sequence $\{(-1)^n\}$ is not a Cauchy sequence.

Let $u_n = (-1)^n$.

Then $|u_m - u_n| = |(-1)^m - (-1)^n|$

$|u_m - u_n| = 0$ if m and n are both even or both odd,

$|u_m - u_n| = 2$ if one of m, n is odd and the other is even.

Let us choose $\varepsilon = 1$. Then it is not possible to find a natural number k such that $|u_m - u_n| < \varepsilon$ for all $m, n \geq k$.

Hence $\{u_n\}$ is not a Cauchy sequence.

Ex 17: Discuss the convergence of $\left\{\frac{n-1}{2n}\right\}$ **VU'2009**

Hints: $|u_{n+p} - u_n| = \frac{p}{2n(n+p)} < \frac{1}{2n}$

Ex 18: Prove that the sequence $\{u_n\}$ where $u_1 = 0, u_2 = 1$ and $u_{n+2} = \frac{1}{2}(u_{n+1} + u_n)$ for all $n \geq 1$ is a Cauchy sequence.

$$u_{n+2} - u_{n+1} = \frac{1}{2}(u_{n+1} + u_n) - u_{n+1} = -\frac{1}{2}(u_{n+1} - u_n)$$

Or, $|u_{n+2} - u_{n+1}| = \frac{1}{2}|u_{n+1} - u_n| = \frac{1}{2^2}|u_n - u_{n-1}| = \ldots\ldots = \frac{1}{2^n}|u_2 - u_1| = \frac{1}{2^n}$

Let $m > n$. Then

$$|u_m - u_n| \leq |u_m - u_{m-1}| + |u_{m-1} - u_{m-2}| + \ldots\ldots + |u_{n+1} - u_n|$$

$$= \left(\frac{1}{2}\right)^{m-2} + \left(\frac{1}{2}\right)^{m-3} + \ldots\ldots + \left(\frac{1}{2}\right)^{n-1}$$

$$= \frac{4}{2^n}\left[1 - \left(\frac{1}{2}\right)^{m-n}\right] < \frac{4}{2^n}$$

Let $\varepsilon > 0$. Then there exists a natural number k such that $\frac{4}{2^n} < \varepsilon$ for all $n \geq k$

Hence $|u_m - u_n| < \varepsilon$ for all $m, n \geq k$

$\Rightarrow$ This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

Ex 19: Prove that the sequence $\{u_n\}$ satisfying the condition $|u_{n+2} - u_{n+1}| \leq c|u_{n+1} - u_n|$ for all $n \in \mathbb{N}$, where $0 < c < 1$ is a Cauchy sequence.

$$|u_{n+2} - u_{n+1}| \leq c|u_{n+1} - u_n| \leq c^2|u_n - u_{n-1}| \leq \ldots\ldots \leq c^n|u_2 - u_1|$$

Let $m > n$. Then

$$|u_m - u_n| \leq |u_m - u_{m-1}| + |u_{m-1} - u_{m-2}| + \ldots\ldots + |u_{n+1} - u_n|$$

$$\leq |u_2 - u_1|\{c^{m-2} + c^{m-3} + \ldots.. + c^{n-1}\}$$

$$= |u_2 - u_1|c^{n-1}\frac{1 - c^{m-n}}{1 - c} < \frac{c^{n-1}}{1 - c}|u_2 - u_1|$$

Let $\varepsilon > 0$. Since $0 < c < 1$, the sequence $\{c^{n-1}\}$ is a convergent sequence. Therefore there exists a natural number k such that $c^{n-1} < \frac{1 - c}{|u_2 - u_1|}\varepsilon$ for all $n \geq k$

Hence $|u_m - u_n| < \varepsilon$ for all $m, n \geq k$

$\Rightarrow$ This proves that the sequence $\{u_n\}$ is a Cauchy sequence.

Ex 20: Let $u_1 = 2$ and $u_{n+1} = 2 + \frac{1}{u_n}$ for $n \ge 1$. Prove that the sequence $\{u_n\}$ converge to the limit $\sqrt{2}+1$

Hints: Clearly $\{u_n\}$ is a sequence of +ve real numbers and $u_n > 2$ for all $n > 1$.

Now $|u_{n+2} - u_{n+1}| = \left|\frac{1}{u_{n+1}} - \frac{1}{u_n}\right| = \frac{|u_{n+1} - u_n|}{u_{n+1}u_n} < \frac{1}{4}|u_{n+1} - u_n|$

Proceeding similar as the previous problem show that $\{u_n\}$ is a Cauchy sequence

$\Rightarrow \{u_n\}$ is convergent

Let $\lim_{n\to\infty} u_n = l$. From the given relation $\lim_{n\to\infty} u_{n+1} = 2 + \lim_{n\to\infty} \frac{1}{u_n}$

$\Rightarrow l = 2 + \frac{1}{l}$ This gives $l = 1 \pm \sqrt{2}$

Since $\{u_n\}$ is a sequence of +ve real numbers $l \ne 1 - \sqrt{2} \Rightarrow l = 1 + \sqrt{2}$

Ex 21: Let $u_1 > 0$ and $u_{n+1} = \frac{1}{2+u_n}$ for $n \ge 1$. Prove that the sequence $\{u_n\}$ converges to the limit $\sqrt{2}-1$

(H.W.)

Ex 22: Let $\{u_n\}$ is a Cauchy sequence in $\mathbb{R}$ having a sub-sequence converging to a real number l, prove that $\lim_{n\to\infty} u_n = l$

Since $\{u_n\}$ is a Cauchy sequence in $\mathbb{R}$ it is convergent

$\Rightarrow$ Every sub-sequence of $\{u_n\}$ converge to the limit of this sequence

Since a sub-sequence of $\{u_n\}$ converge to $l \Rightarrow \lim_{n\to\infty} u_n = l$

Ex 23: Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$ and $\{y_n\}$ is a sequence in $\mathbb{R}$ such that $|x_n - y_n| < \frac{1}{n}$ for all $n \ge 1$.

Prove that $\{y_n\}$ is a Cauchy sequence and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$

Since $\{x_n\}$ be a Cauchy sequence in $\mathbb{R} \Rightarrow \{x_n\}$ is convergent.

Let $\varepsilon > 0$ then by Archimedean property there exists a natural number k such that $k\varepsilon > 1 \Rightarrow \frac{1}{k} < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon$ for all $n \ge k$

Thus for every +ve ε there exists a natural number k such that $|(x_n - y_n) - 0| < \varepsilon$ for all $n \ge k$

$\Rightarrow \lim_{n\to\infty}(x_n - y_n) = 0 \Rightarrow \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$

$\Rightarrow \{y_n\}$ is convergent in $\mathbb{R}$

$\Rightarrow \{y_n\}$ is a Cauchy sequence

Theorem (Cauchy's theorem on limits): If $\lim_{n\to\infty} u_n = l$ then $\lim_{n\to\infty} \frac{u_1 + u_2 + \dots\dots\dots + u_n}{n} = l$.

VU'1998, 04, 06, 07, CU'1999, 01

Proof: Case1: $l = 0$.

Since $\{u_n\}$ is a convergent sequence, it is bounded. Therefore there exists a positive number B such that $|u_n| < B$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} u_n = 0$, there exists a natural number k_1 such that $|u_n| < \frac{\varepsilon}{2}$ for all $n \geq k_1$

Now $\left|\frac{u_1 + u_2 + \dots\dots\dots + u_n}{n}\right| \leq \left|\frac{u_1 + u_2 + \dots\dots\dots + u_{k_1-1}}{n}\right| + \left|\frac{u_{k_1} + u_{k_1+1} + \dots\dots\dots + u_n}{n}\right|$

$$\leq \left|\frac{|u_1| + |u_2| + \dots\dots\dots + |u_{k_1-1}|}{n}\right| + \left|\frac{|u_{k_1}| + |u_{k_1+1}| + \dots\dots\dots\dots + |u_n|}{n}\right|$$

$$< \frac{B(k_1 - 1)}{n} + \frac{n - k_1 + 1}{n} \cdot \frac{\varepsilon}{2} \text{ for all } n \geq k_1.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, there exists a natural number k_2 such that $\left|\frac{1}{n} - 0\right| < \frac{\varepsilon}{2Bk_1}$ for all $n \geq k_2$

I.e. $\frac{Bk_1}{n} < \frac{\varepsilon}{2}$ for all $n \geq k_2$

Let $k = \max\{k_1, k_2\}$. Then $\left|\frac{u_1 + u_2 + \dots\dots\dots + u_n}{n}\right| < \varepsilon$ for all $n \geq k$

This proves that $\lim_{n\to\infty} \frac{u_1 + u_2 + \dots\dots\dots + u_n}{n} = 0$.

Case2: $l \neq 0$

Let $v_n = u_n - l$. Then $\lim_{n\to\infty} v_n = 0$

Now $\frac{u_1 + u_2 + \dots\dots\dots + u_n}{n} - l = \frac{v_1 + v_2 + \dots\dots\dots + v_n}{n}$

By case1, $\lim_{n\to\infty} \frac{v_1 + v_2 + \dots\dots\dots + v_n}{n} = 0$

Therefore $\lim_{n\to\infty} \frac{u_1 + u_2 + \dots\dots\dots + u_n}{n} = l$

Note: The converse of the theorem is not true. Let us consider the sequence $\{u_n\}$ where $u_n = (-1)^n$.

Then $\lim_{n\to\infty} \frac{u_1 + u_2 + \dots\dots\dots + u_n}{n} = 0$ but the sequence $\{u_n\}$ is not convergent.

Corollary: If $\lim_{n\to\infty} u_n = l$ where $u_n > 0$ for all n and $l \neq 0$, then $\lim_{n\to\infty} \sqrt[n]{u_1 u_2 \dots\dots\dots u_n} = l$.

Proof: Let $v_n = \log u_n$.

Since each u_n is positive and $\lim_{n\to\infty} u_n = l > 0$, the sequence $\{v_n\}$ converges to $\log l$.

By Cauchy's theorem on limit we have,

$$\lim_{n\to\infty} \frac{v_1 + v_2 + \dots\dots\dots + v_n}{n} = \log l.$$

Or, $\lim_{n\to\infty} \frac{\log u_1 + \log u_2 + \dots\dots\dots + \log u_n}{n} = \log l$

Or, $\lim_{n\to\infty} \log \sqrt[n]{u_1 u_2 \dots\dots\dots u_n} = \log l$

It follows that $\lim_{n\to\infty} \sqrt[n]{u_1 u_2 \dots\dots\dots u_n} = l$.

Cesaro's theorem: If the sequences $\{a_n\}$ and $\{b_n\}$ converges to finite limits a and b respectively, then $\lim_{n\to\infty}\frac{a_1b_n+a_2b_{n-1}+......+a_nb_1}{n}=ab$ **VU'2003**

Proof: Let $a_n=a+\alpha_n$ where $|\alpha_n|\to 0$ as $n\to\infty$

Substituting the values of $a_1,a_2,...,a_n$ we get

$$\frac{a_1b_n+a_2b_{n-1}+......+a_nb_1}{n}=\frac{a(b_1+b_2+...+b_n)}{n}+\frac{\alpha_1b_n+\alpha_2b_{n-1}+...+\alpha_nb_1}{n}$$

$$\Rightarrow \lim_{n\to\infty}\frac{a_1b_n+a_2b_{n-1}+......+a_nb_1}{n}=\lim_{n\to\infty}\frac{a(b_1+b_2+...+b_n)}{n}+\lim_{n\to\infty}\frac{\alpha_1b_n+\alpha_2b_{n-1}+...+\alpha_nb_1}{n}$$

.... (i)

Since $\{b_n\}$ converges to b therefore $\{b_n\}$ is bounded i.e. there exists a +ve real number B such that $|b_n|\le B$ for all $n\in\mathbb{N}$ also by Cauchy's limit theorem $\lim_{n\to\infty}\frac{b_1+b_2+....+b_n}{n}=b\Rightarrow$

$$\lim_{n\to\infty}\frac{a(b_1+b_2+...+b_n)}{n}=ab \text{ ....(ii)}$$

$$\left|\frac{\alpha_1b_n+\alpha_2b_{n-1}+...+\alpha_nb_1}{n}\right|\le\frac{B(|\alpha_1|+|\alpha_2|+.....+|\alpha_n|)}{n} \text{ .... (iii)}$$

Now since $|\alpha_n|\to 0$ as $n\to\infty$ then by Cauchy's limit theorem $\lim_{n\to\infty}\frac{|\alpha_1|+|\alpha_2|+.....+|\alpha_n|}{n}=0$

From (iii) we get $\lim_{n\to\infty}\frac{\alpha_1b_n+\alpha_2b_{n-1}+...+\alpha_nb_1}{n}=0$ .... (iv)

From (i), (ii) and (iv) we get $\lim_{n\to\infty}\frac{a_1b_n+a_2b_{n-1}+......+a_nb_1}{n}=ab$

Ex 24: Prove that $\lim_{n\to\infty}\frac{1+\frac{1}{2}+..........+\frac{1}{n}}{n}=0$.

Let $u_n=\frac{1}{n}$. Then $\lim_{n\to\infty}u_n=0$

By Cauchy's theorem, $\lim_{n\to\infty}\frac{1+\frac{1}{2}+..........+\frac{1}{n}}{n}=0$

Ex 25: Prove that $\lim_{n\to\infty}\frac{1+\sqrt{2}+\sqrt[3]{3}+...........+\sqrt[n]{n}}{n}=1$. **VU'1997, CU'2004**

Let $u_n=\sqrt[n]{n}$. Then $\lim_{n\to\infty}u_n=1$

By Cauchy's theorem, $\lim_{n\to\infty}\frac{1+\sqrt{2}+\sqrt[3]{3}+...........+\sqrt[n]{n}}{n}=1$.

Theorem: Let $u_n>0$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=l$ (finite or infinite). Then $\lim_{n\to\infty}\sqrt[n]{u_n}=l$

Ex 26: Prove that $\lim_{n\to\infty}n^{\frac{1}{n}}=1$

Let $u_n = n$. Then $u_n > 0$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 1 \Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{u_n} = 1$ i.e. $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$

Ex 27: Prove that $\lim\limits_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$

Let $u_n = \frac{n!}{n^n}$. Then $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{e}$

$\Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{u_n} = \frac{1}{e}$ i.e. $\lim\limits_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$

Ex 28: Prove that $\lim\limits_{n\to\infty} \frac{\{(n+1)(n+2).......2n\}^{\frac{1}{n}}}{n} = \frac{4}{e}$

Let $u_n = \frac{(n+1)(n+2)....2n}{n^n}$

Then $u_n > 0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{4}{e}$

$\Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{u_n} = \frac{4}{e}$ i.e. $\lim\limits_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{4}{e}$