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Unit - 4 (C9T) for SEM-4 (H)

EX. 4 Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint(xdy - ydx)$.
Hence find the area of the ellipse $x = a\cos\theta$
$y = b\sin\theta$.

Sol. By the Green's Theorem

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where curve C is traversed in the +ve direction. M and N are the continuous functions of x and y ~~in~~ having continuous derivative in R.

Now in Green's Theorem put $M = -y$ & $N = x$ then

$$\oint_C (xdy - ydx) = \iint_R \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\, dx\,dy$$

$$= 2\iint_R dx\,dy = 2A.$$ where A is the required area.

$$\therefore A = \frac{1}{2}\oint_C (xdy - ydx)$$

[by $x = a\cos\theta$
$y = b\sin\theta$
$dx = -a\sin\theta\, d\theta$
$dy = b\cos\theta\, d\theta$.
θ ranges from 0 to 2π]

$$\therefore A = \frac{1}{2}\int_0^{2\pi}\left\{(a\cos\theta)\times(b\cos\theta\, d\theta) - (b\sin\theta)(-a\sin\theta\, d\theta)\right\}$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)\,d\theta$$

$$= \frac{ab}{2}\left[\theta\right]_0^{2\pi} = \frac{ab}{2}\cdot 2\pi = ab\pi.$$

$$= \pi ab.$$

Ex 5. Evaluate the surface integral $\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds$ by transforming it into a line integral. S being that part of surface of the Paraboloid $z = 1 - x^2 - y^2$ for which $z \geq 0$ and
$F = y\hat{i} + z\hat{j} + x\hat{k}$

Sol^n: for the xy-plane $z=0$, or $x^2+y^2=1$ the circle of radius 1

It is Stokes' theorem the relation between surface integral and line integral

i.e. $\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds = \oint \vec{F}\cdot d\vec{r} = \cancel{\oint (y\hat{i}+z\hat{j}+x\hat{k})\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})} = \cancel{y+z+x}$

Now $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$

$= \hat{i}\left\{\frac{\partial}{\partial y}(x) - \frac{\partial}{\partial z}(z)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial z}(y)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial y}(y)\right\}$

$= -\hat{i} - \hat{j} - \hat{k}$

here obvious $\hat{n} = \hat{k}$.

$\therefore \iint_S (\nabla\times F)\cdot\hat{n}\,ds = \iint_S k(-1)\cdot k\,dx\,dy = -\iint_S dx\,dy$

which is area of the circle of radius $r=1$

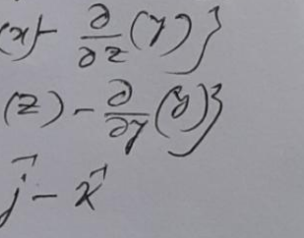
so ~~area~~ $= -\pi(1)^2 = -\pi$

Ex 6 Verify Green's Theorem in plane for $\oint[(3x^2-8y^2)dx + (4y-6xy)dy]$ where C is the boundary of the region bounded by $x \geq 0$, $y \leq 0$, and $2x - 3y = 6$.

Hints: By Green's Th. $\oint(M\,dx + N\,dy)$ Here $M = 3x^2 - 8y^2$, $N = 4y - 6xy$.

$= \iint_R \left[\frac{\partial}{\partial x}(N) - \frac{\partial}{\partial y}(M)\right]dx\,dy$

$= \iint_R 10y\,dx\,dy = 10\int_0^2 dx \int_{\frac{1}{3}(2x-6)}^{0} y\,dy$ = do it complete $= -20$.

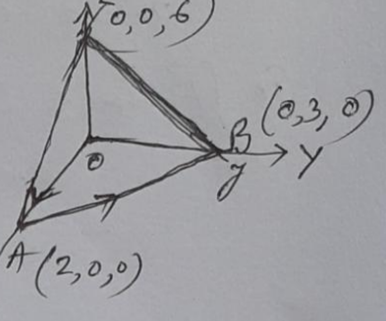


Now evaluate line integral along OB, $x=0$, $dx=0$, y ranges from 0 to -2

Along BA, $x = \frac{1}{2}(6+3y)$, $dx = \frac{3}{2}dy$ and y varies from $-2, 0$

Along AO, $y=0$, $dy=0$ and x varies from 3 to 0 find the value it is -20 (Proved)

EX(7) Verify Stoke's Theorem for $\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$ for the surface of triangular lamina with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$



Solⁿ: Here the path integration along AB, BC, CA, in the fig. where co-ordinates of $A(2,0,0)$, $B(0,3,0)$ and $C(0,0,6)$. Let S be the plane surface of the triangle ABC bounded by curve C. Let $\hat{n}$ be unit normal vector to surface S. and equation of plane is

$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$. Now Stokes Theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,ds \quad \text{——— (1)}$$

Hence $\oint_C \vec{F}\cdot d\vec{r} = \int_{ABC} \vec{F}\cdot d\vec{r} = \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BC}\vec{F}\cdot d\vec{r} + \int_{CA}\vec{F}\cdot d\vec{r}$

Along line AB, $z=0$, the equⁿ of AB is $\frac{x}{2} + \frac{y}{3} = 1 \Rightarrow y = \frac{3}{2}(2-x)$

$\therefore dy = -\frac{3}{2}dx$ x varies from 2 to 0

So $\int_{AB}\vec{F}\cdot d\vec{r} = \int_{AB}[(x+y)\hat{i} + 2x\hat{j} + y\hat{k}]\cdot[\hat{i}dx + \hat{j}dy]$ $\quad r = \hat{i}x + \hat{j}y$

$= \cancel{\int_2^0 f\,dx} + \int_{AB}[(x+y)dx + 2x\,dy]$

$= \int_2^0 \{[x + \frac{3}{2}(2-x)]\cdot dx + 2x(-\frac{3}{2})dx\} = \int_0^2 (-\frac{7x}{2} + 3)dx$

$= \left[-7\frac{x^2}{4} + 3x\right]_0^2 = 7 - 6 = +1$ ——— (A)

Along line BC, $x=0$, equⁿ of BC is $\frac{y}{3} + \frac{z}{6} = 1$, or $z = 6-2y$
y ranges from 3 to 0. $\quad dz = -2dy$
Here $r = y\hat{j} + z\hat{k}$.

$\therefore \int_{BC} F\cdot dr = \int_{BC}\{(y\hat{i} - z\hat{j} + (y+z)dz\}\cdot(\hat{j}dy + \hat{k}dz) = \int_{BC}[-z\,dy + (y+z)dz]$

$= \int_3^0 [(2y-6)dy + (y+6-2y)(-2dy)]$

$= \int_3^0 (4y-18)dy = \left[2y^2 - 18y\right]_3^0 = 36$. ——— (B)

Next

Along line CA, $y=0$, Equ^n. CA $\frac{x}{2}+\frac{z}{6}=1$

or. $z = 6-3x$, $dz = -3dx$

x varies from 0 to 2 and $r = x\hat{i} + z\hat{k}$

So $\int_{CA} \vec{F}\cdot d\vec{r} = \int_{CA} [x i + (2x-z) j + z\hat{k}] \cdot \cancel{[dx\hat{i}+dz\hat{k}]}$
$\cdot [dx\hat{i} + dz\hat{k}]$

$= \int_{CA} (x\,dx + z\,dz) = \int_0^2 [x\,dx + (6-3x)(-3dx)]$

$= \int_0^2 (10x-18)dx = \left[5x^2-18x\right]_0^2 = -16$ — (C)

Hence L.H.S. of (1) $\int_{ABC} \vec{F}\cdot d\vec{r} = 1 + 36 - 16 = 21$ - . (2)
(A)+(B)+(C)

Now $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = (1+1)i - (0-0)j + (2-1)\hat{k} = 2\hat{i} + \hat{k}$.

Now normal to ABC (Plane) $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$

is $\nabla\phi = \left\{\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right\}\left\{\frac{x}{2}+\frac{y}{3}+\frac{z}{6}-1\right\}$

$= \frac{\hat{i}}{2} + \frac{\hat{j}}{3} + \frac{\hat{k}}{6}$

Normal unit vector $\hat{n} = \dfrac{\frac{\hat{i}}{2}+\frac{\hat{j}}{3}+\frac{\hat{k}}{6}}{\sqrt{\frac{1}{4}+\frac{1}{9}+\frac{1}{36}}} = \frac{1}{\sqrt{14}}(3\hat{i}+2\hat{j}+\hat{k})$

hence RHS of (1) $\iint_S (\nabla\times F)\cdot\hat{n}\,dS = \iint_S (2\hat{i}+\hat{k})\cdot\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}} \dfrac{dx\,dy}{\frac{1}{\sqrt{14}}(3\hat{i}+2\hat{j}+\hat{k})\cdot\hat{k}}$

$= \iint_S \dfrac{(6+1)\,dx\,dy}{\sqrt{14}\cdot\frac{1}{\sqrt{14}}}$ Here $\left[dS = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}}\right.$.

$= 7\iint dx\,dy = 7$ Area of triangle OAB

$= 7\times\left(\frac{1}{2}\times 2\times 3\right) = 21$

Hence Stoke's theorem is verified.