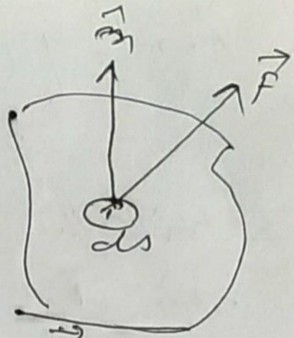
4th Sem 28/2/19 Surface integral — (1) Notes on

SURFCE INTEGRAL

Surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.



∴ Component of $\vec{F}$ along the normal $= \vec{F} \cdot \vec{n}$ where $\hat{n}$ is the unit normal vector to an element ds where $\vec{n} = \dfrac{\text{grad} f}{|\text{grad} f|}$, $ds = \dfrac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$

hence Surface integral over S $= \iint_S (\vec{F} \cdot \vec{n})\, ds$

clearly $\vec{ds} = \vec{n}\, ds = dy\,dz\,\hat{i} + dz\,dx\,\hat{j} + dx\,dy\,\hat{k}$

if R is the Projection of S ~~on the~~

then $\iint_S \vec{F} \cdot \vec{n}\, ds = \iint_R \vec{F} \cdot \vec{n} \dfrac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$, R is the Projection of S on xy-plane

$= \iint_R \vec{F} \cdot \vec{n} \dfrac{dy\,dz}{|\vec{n} \cdot \hat{i}|} = \iint_R \vec{F} \cdot \hat{n} \dfrac{dz\,dx}{|\vec{n} \cdot j|}$

⇓ R is the Projection of S on yz plane

⇓ R is the Projection of S on zx plane

Ex. Evaluate $\iint_S \vec{A} \cdot \vec{n}\, ds$ where $\vec{A} = (x + y^2)\,i - 2xj + 2yz\,\vec{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the 1st octant.

Solⁿ. A vector normal to surface 'S' is given by

$$\nabla(2x + y + 2z) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(2x + y + 2z) = 2i + j + 2k$$

And unit vector to normal surface s is $\hat{n}$

$$= \frac{2i + j + 2k}{\sqrt{4 + 1 + 4}} = \frac{2}{3}i + \frac{1}{3}j + \frac{2}{3}k.$$

now $\vec{n} \cdot \hat{k} = \hat{k}\left(\frac{2}{3}i + \frac{1}{3}j + \frac{2}{3}k\right) = \frac{2}{3}$

We know that

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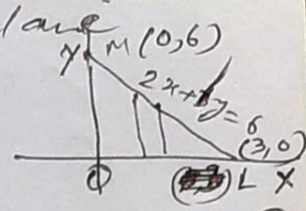
$$\iint_S \vec{A}\cdot\vec{n}\,ds = \iint_R \vec{A}\cdot\vec{n}\,\frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$$ where R is the projection ~~on~~ projection on ~~xy plane~~ xy plane, $z=0$.

Now $\vec{A}\cdot\vec{n} = \left[(x+y^2)\hat{i} - 2x\hat{j} + 2yz\vec{k}\right]\cdot\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\vec{k}\right)$

$$= \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}yz \quad \text{--- (1)}$$

Putting the value of $z = \frac{6-\cancel{2x}-y)}{2}$ from plane in (1) we get



$$\vec{A}\cdot\vec{n} = \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right)$$

$$= \frac{2}{3}y(y+6-2x-y) = \frac{4}{3}y(3-x) \quad \text{--- (2)}$$

Hence $$\iint_S \vec{A}\cdot\vec{n}\,ds = \iint_R \vec{A}\cdot\vec{n}\,\frac{dx\,dy}{|\vec{n}\cdot\vec{k}|} \quad \text{--- (3)}$$

Putting the values of $\vec{A}\cdot\vec{n}$ from (2) in (3) we get:

[where R is the projection of S]

$$\iint_S \vec{A}\cdot\vec{n}\,ds = \iint_R \frac{4}{3}y(3-x)\frac{3}{2}\,dx\,dy$$

$$= \int_0^3\int_0^{6-2x} 2y(3-x)\,dx\,dy = \int_0^3 2(3-x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

$$= \int_0^3 (3-x)(6-2x)^2\,dx = 4\int_0^3 (3-x)^3\,dx$$

$$= 4\left[\frac{(3-x)^4}{4(-1)}\right]_0^3 = -[0-81] = 81.$$

<u>EX</u>. Find the surface integral of the vector $\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\vec{k}$ over S, the part of the plane $2x+3y+6z=12$ included in the first Octant or Evaluate $\iint_S \vec{F}\cdot\vec{n}\,ds$ where

$F = 18z\hat{i} - 12\hat{j} + 3y\vec{k}$ and S is the part of the plane $2x+3y+6z=12$ included in the 1st octant

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EX: $\iint_S \vec{A}\cdot\hat{n}\,ds$, $A = 18zi - 12j + 3yk$ and S the plane $2x+3y+6z=12$ in the 1st octant.

Solⁿ: Here $\vec{A} = 18zi - 12j + 3yk$, Given surface

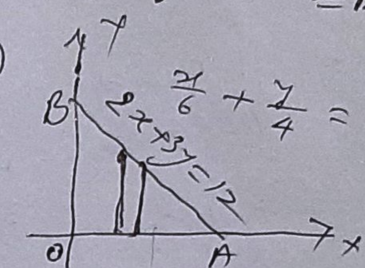
$f(x,y,z) = 2x+3y+6z-12$

∴ Normal Vector $\nabla f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(2x+3y+6z-12)$

$= 2i + 3j + 6\hat{k}$

$\hat{n}$ = the unit Vector at any point (x,y,z) of $2x+3y+6z=12$

$= \frac{2i+3j+6k}{\sqrt{4+9+36}} = \frac{1}{7}(2i+3j+6k)$



$d\vec{S} = \frac{dx\,dy}{|\vec{n}\cdot k|}$ Projection xy plane.

$= \frac{dx\,dy}{\frac{1}{7}(2i+3j+6k)\cdot k} = \frac{7}{6}dx\,dy$

Now $\iint \vec{A}\cdot\hat{n}\,ds = \iint (18zi - 12j + 3yk)\cdot\frac{1}{7}(2i+3j+6k)\frac{7}{6}dx\,dy$

$= \iint (36z - 36 + 18y)\frac{dx\,dy}{6} = \iint (6z - 6 + 3y)\,dx\,dy$

Putting the value of $6z = 12 - 2x - 3y$, we get

$= \int_0^6\int_0^{\frac{12-2x}{3}} (12 - 2x - 3y - 6 + 3y)\,dx\,dy$

$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (6-2x)\,dx\,dy = \int_0^6 (6-2x)\left[y\right]_0^{\frac{1}{3}(12-2x)}$

$= \int_0^6 (6-2x)\cdot\frac{1}{3}(12-2x)\,dx = \frac{1}{3}\int_0^6 (4x^2 - 36x + 72)\,dx$

$= \frac{1}{3}\left[\frac{4x^3}{3} - \frac{36x^2}{2} + 72x\right]_0^6 =$

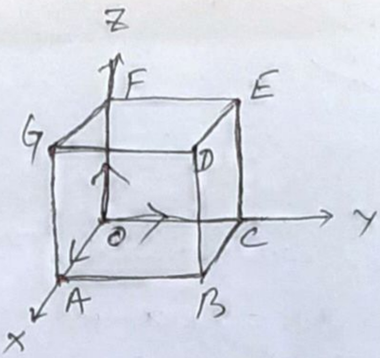
$= \frac{1}{3}\left[\frac{4\times 6\times 6\times 6}{3} - \frac{36\times 6\times 6}{2} + 72\times 6 - 0\right] = \underline{24}$

Ex ② S

Show that $\iint_S \vec{F}\cdot\vec{n}\,ds = \frac{3}{2}$

where $F = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

and S is the surface of the cube bounded by the planes $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$



Surface	Normal	ds
OABC	$-k$	$dx\,dy \quad z=0$
DEFG	k	$dx\,dy \quad z=1$
OAGF	$-j$	$dx\,dz,\ y=0$
BCED	j	$dx\,dz \quad y=1$
ABDG	i	$dy\,dz \quad x=1$
OCEF	$-i$	$dy\,dz,\ x=0$

Sol^n:- $\iint_S F\cdot\vec{n}\,ds = \iint_{OABC} \vec{F}\cdot\vec{n}\,ds + \iint_{DEFG} \vec{F}\cdot\vec{n}\,ds + \iint_{OAGF} \vec{F}\cdot\vec{n}\,ds + \iint_{BCED} \vec{F}\cdot\vec{n}\,ds + \iint_{ABDG} \vec{F}\cdot\vec{n}\,ds + \iint_{OCEF} \vec{F}\cdot\vec{n}\,ds$ ——— ①

Now $\iint_{OABC} \vec{F}\cdot\vec{n}\,ds = \iint_{OABC} (4xz\hat{i} - y^2\hat{j} + yz\vec{k})\cdot(-k)\,dx\,dy = \int_0^1\int_0^1 -yz\,dx\,dy = 0$ as $z=0$.

$\iint_{DEFG} (4xz\hat{i} - y^2\hat{j} + yz\vec{k})\cdot(k)\,dx\,dy) = \iint_{DEFG} yz\,dx\,dy = \int_0^1\int_0^1 y.1\,dx\,dy$. as $z=1$

$= \int_0^1 dx \left[\frac{y^2}{2}\right]_0^1 = [x]_0^1 = \frac{1}{2}$

$\iint_{OAGF} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{j})\,dx\,dz = \iint_{OAGF} y^2\,dx\,dz = 0$ as $y=0$

$\iint_{BCED} (4xz\hat{i} - y^2\hat{j} + yz\vec{k})\cdot(\hat{j})\,dx\,dz = \iint_{BCED} -y^2\,dx\,dz = -\int_0^1 dx \int_0^1 dz \qquad y=1$

$= -[x]_0^1 = -1$

$\iint_{ABDG} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(\hat{i})\,dy\,dz = \iint 4xz\,dy\,dz = \int_0^1\int_0^1 4.1.z\,dy\,dz$, as $x=1$

$= 4[y]_0^1 \left[\frac{z^2}{2}\right]_0^1 = 4\cdot\frac{1}{2} = 2$

$\iint_{OCEF} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{i})\,dy\,dz = \int_0^1\int_0^1 -4\cancel{x}z\,dy\,dz = 0$ as $x=0$.

Hence $\iint_S \vec{F}\cdot\vec{n}\,ds = 0 + \frac{1}{2} + 0 - 1 + 2 + 0 = \frac{3}{2}$

NB: In $\vec{F}$ is is a conservative field and $F = \nabla\phi$, then ϕ is called the scalar potential of the field F.

x. Evaluate $\iint_S (yz\hat{i}+zx\hat{j}+xy\hat{k})\cdot ds$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ in the first octant

Sol^n:- Here $\phi = x^2+y^2+z^2-a^2$

Normal vector to surface $= \nabla\phi =$

$$= \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-a^2)$$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{4(x^2+y^2+z^2)}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

as $x^2+y^2+z^2=a^2$

Here $\vec{F} = yz\hat{i}+zx\hat{j}+xy\hat{k}$

$$\therefore \vec{F}\cdot\hat{n} = (yz\hat{i}+zx\hat{j}+xy\hat{k})\cdot\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

$$= \frac{3xyz}{a}$$

Now ~~$\iint_S \vec{F}\cdot\hat{n}$~~ $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S (\vec{F}\cdot\hat{n})\frac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$

$$= \int_0^a\int_0^{\sqrt{a^2-x^2}} \frac{3xyz\,dx\,dy}{a\left(\frac{z}{a}\right)}$$

Projection on xy plane.

$$= 3\int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= 3\int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{3}{2}\int_0^a x(a^2-x^2)\,dx = \frac{3}{2}\left[\frac{a^2x^2}{2}-\frac{x^4}{4}\right]_0^a$$

$$= \frac{3}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right] = \frac{3a^2}{8}.$$