Topic - Differential Equation
(System of Linear differential Equation) F039870

SEM - 02, Unit - 2, CC - 4.

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Introduction - Let x, y be two independent variables and t be the independent variables. Then the equations will involve derivatives of x, y w.r.t t. We denote the operator $\frac{d}{dt}$ by D. Then such equation will be of the form

$$f_1(D)x + f_2(D)y = T_1 \quad \text{———} (1) \quad \&$$

$$\phi_1(D)x + \phi_2(D)y = T_2 \quad \text{———} (2)$$

where f_1, f_2, ϕ_1, ϕ_2 are all rational functions of D with constant co-efficients and T_1, T_2 are functions of independent variables t.

Methods of Solutions: There are two formal methods of solutions for such types of equations. They are (1) Symbolic method or method of elimination & (2) Method of differentiation.

The following examples will illustrate the process.

Ex.1. Solve.

$$\frac{dx}{dt} + 5x + y = e^t \quad \text{———} (1)$$

$$\frac{dy}{dt} - x + 3y = e^{2t}. \quad \text{———} (2).$$

Ans: We solve the problem by the method of differentiation.

Differentiating (1) w.r.t t we get

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + \frac{dy}{dt} = e^t \quad \text{———} (3)$$

From (1) we get, $\quad y = e^t - 5x - \frac{dx}{dt}$. ——— (4)

Putting the value of y in (2), we get

$$\frac{dy}{dt} - x + 3e^t - 15x - 3\frac{dx}{dt} = e^{2t}$$

or, $$\frac{dy}{dt} = 16x - 3e^t + 3\frac{dx}{dt} + e^{2t}$$

Putting the above values of $\frac{dy}{dt}$ in (3), we get

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 16x - 3e^t + 3\frac{dx}{dt} + e^{2t} = e^t$$

or $$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 4e^t - e^{2t} \quad (5)$$

or $$(D^2 + 8D + 16)x = 4e^t - e^{2t}$$

A.E. of of homogeneous part of (5) is

$$m^2 + 8m + 16 = 0 \quad \Rightarrow m = -4, -4.$$

$\therefore$ C.F $= (C_1 + C_2 t)e^{-4t}$. $\quad C_1, C_2$ are A.C.

$$P.I = \frac{4}{(D+4)^2}e^t - \frac{1}{(D+4)^2}e^{2t}$$

$$= \frac{4}{25}e^t - \frac{e^{2t}}{36}.$$

$$\therefore x = (C_1 + C_2 t)e^{-4t} + \frac{4}{25}e^t - \frac{e^{2t}}{36} \quad \text{———} (I)$$

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- Again $\frac{dx}{dt} = -5x - y + e^{t}$ [From (1)] Page-3

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differentiating (I) w.r.t t we get

$$\frac{dx}{dt} = -4(c_1 + c_2 t)\bar{e}^{4t} + c_2\,\bar{e}^{4t} + \frac{4}{25}e^{t} - \frac{e^{2t}}{18} \qquad -⑥$$

Using (I) & ⑥ in ④ we get

$$y = -(c_1 + c_2 + c_2 t)\,\bar{e}^{4t} + \frac{7}{36}e^{2t} + \frac{e^{t}}{25} \qquad \text{—— (II)}.$$

(I) & (II) gives the required general solutions of the given system of equation.

Ex 2. Solve $\frac{d^2x}{dt^2} - 3x - 4y = 0$ —— (1)

$\frac{d^2y}{dt^2} + x + y = 0$ —— (2)

- We use symbolic method; The given equations can be written as

$$(D^2-3)x - 4y = 0 \qquad \text{—— (3)}$$

$$(D^2+1)y + x = 0 \qquad \text{—— (4)}$$

Eleminating y, operating (D^2+1) on (3) and Multiplying (4) by -4 and then Subtracting we get

$$(D^2+1)(D^2-3)x + 4x = 0$$

$$(D^2-1)^2 x = 0$$

This is a linear differential equation with Constan coefficient. So we solve it for x.

A.E is $(m^2-1)^2 = 0 \Rightarrow m = \pm 1$.

$\therefore\ x = (c_1 + c_2 t)e^{t} + (c_3 + c_4 t)e^{-t}$ c_1, c_2, c_3, c_4 A.C

— (5). (I)

differentiating (5) we get

$$\frac{dx}{dt} = (c_1 + c_2 t)e^{t} + (c_3 + c_4 t)e^{-t}(-1) + c_2 e^{t} + c_4 e^{-t}$$

$$\frac{d^2x}{dt^2} = (c_1 + c_2 t)e^{t} + (c_3 + c_4 t)e^{-t} + c_2 e^{t} - c_4 e^{-t} + c_2 e^{t} + c_4 e^{-t}(-1).$$

— (6)

Using (5) & (6) in (1) we get

$$(c_1 + c_2 t)e^{t} + (c_3 + c_4 t)e^{-t} + 2c_2 e^{t} - 2c_4 e^{-t} - 3(c_1 + c_2 t)e^{t} - 3(c_3 + c_4 t)e^{-t} = 4y$$

$$\therefore\ 2y = -(c_1 + c_2 t)e^{t} - (c_3 + c_4 t)e^{-t} + c_2 e^{t} - c_4 e^{-t}$$

$$\therefore\ y = \frac{1}{2}e^{t}(c_2 - c_1 - c_2 t) - \frac{1}{2}e^{-t}(c_3 + c_4 t + c_4) \quad \text{— (II)}$$

(I) & (II) together gives the required G.S.

Ex 3. Solve

$$\frac{dx}{dt} = 7x - y \quad \text{— (1)}$$

$$\frac{dy}{dt} = 2x + 5y \quad \text{— (2)}$$

Ans. The equations can be written as

$$(D-7)x + y = 0 \quad \text{— (3)}$$

$$-2x + (D-5)y = 0 \quad \text{— (4)}$$

Eleminating y between (3) and (4) we get

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$$(D-5)(D-7)x + 2x = 0$$

or $(D^2 - 12D + 37)x = 0$

A.E is $m^2 - 12m + 37 = 0 \Rightarrow m = 6 \pm i$

$\therefore x = e^{6t}(c_1 \cos t + c_2 \sin t)$ —— (I) c_1, c_2 are A.C.

differentiating w.r.t. t we get

$$\frac{dx}{dt} = 6e^{6t}(c_1 \cos t + c_2 \sin t) + e^{6t}(-c_1 \sin t + c_2 \cos t) \quad \text{—— (5)}$$

using (I) & (5) in (1) we get

$$y = e^{6t}\{(c_1 - c_2)\cos t + (c_1 + c_2)\sin t\}. \quad \text{—— (II)}$$

(I) & (II) together gives the required G.S. of (1) & (2)

Q.4 Solve $\frac{dx}{dt} + 4x + 3y = t$ —— (1)

$\frac{dy}{dt} + 2x + 5y = e^t$. —— (2)

Ans The given equations can be written as

$(D+4)x + 3y = t$ —— (3)

$2x + (D+5)y = e^t$ —— (4)

Eleminating (3) & (4) we get

$$(D+4)(D+5)x - 6x = (D+5)t - 3e^t$$

or $(D^2 + 9D + 14)x = \frac{d}{dt}(t) + 5t - 3e^t = 1 + 5t - 3e^t$

A.E is $m^2 + 9m + 14 = 0 \Rightarrow m = -7, -2$

$\therefore$ C.F. $= c_1 e^{-7t} + c_2 e^{-2t}$.

Also P.I $= \frac{1}{(D+7)(D+2)}(1 + 5t - 3e^t)$

$$= \frac{1}{14} + 5 \cdot \frac{1}{(D+7)(2+D)} t - 3\frac{e^t}{24}$$

$$= \frac{1}{14} + \frac{5}{14} \cdot \left\{ \left(1+\frac{D}{7}\right)^{-1} \left(1+\frac{D}{2}\right)^{-1} \right\} t - \frac{e^t}{8}$$

$$= \frac{1}{14} + \frac{5}{14} \left\{ \left(1-\frac{D}{7}\right)\left(1-\frac{D}{2}\right) \right\} t - \frac{e^t}{8}$$

$$= \frac{1}{14} + \frac{5}{14} \left\{ \left(1 - \frac{9D}{14}\right) t \right\} - \frac{e^t}{8}$$

$$= \frac{1}{14} + \frac{5t}{14} - \frac{5}{14} \cdot \frac{9}{14} - \frac{e^t}{8} = \frac{5t}{14} - \frac{31}{196} - \frac{e^t}{8}.$$

$$\therefore x = c_1 e^{-7t} + c_2 e^{-2t} + \frac{5t}{14} - \frac{e^t}{8} - \frac{31}{196}. \quad \text{——(I)}$$

From (1) we get

$$3y = t - 4x - \left(\frac{dx}{dt}\right)$$

$$= t - 4x - \left(-7c_1 e^{-7t} - 2c_2 e^{-2t} + \frac{5}{14} - \frac{e^t}{8}\right)$$

$$= t - 4c_1 e^{-7t} - 4c_2 e^{-2t} - \frac{20t}{14} + \frac{31}{49} + \frac{e^t}{2}$$
$$+ 7c_1 e^{-7t} + 2c_2 e^{-2t} - \frac{5}{14} + \frac{e^t}{8}.$$

$$\therefore y = \frac{1}{3}\left[-\frac{3}{7}t + 3c_1 e^{-7t} - 2c_2 e^{-2t} + \frac{27}{98} + \frac{5e^t}{8} \right]$$

——(II)

(I) & (II) together gives the required soln.

Ex5 Solve $\frac{dx}{dt} + \frac{2}{t}(x-y) = 1$ ——— (1)

$\frac{dy}{dt} + \frac{1}{t}(x+5y) = t$ ——— (2)

A. The equations can be written as

$t\frac{dx}{dt} + 2x - 2y = t$ ——— (3)

$t\frac{dy}{dt} + x + 5y = t^2$ ——— (4)

differentiating (3) w.r.t 't' we get

$t\frac{d^2x}{dt^2} + \frac{dx}{dt} + 2\frac{dx}{dt} - 2\frac{dy}{dt} = 1$ ——— (5)

Using (4) in (5) we get

$t\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - \frac{2}{t}(t^2 - x - 5y) = 1$

or $t^2\frac{d^2x}{dt^2} + 3t\frac{dx}{dt} - 2t^2 + 2x + 10y = t$

or $t^2\frac{d^2x}{dt^2} + 3t\frac{dx}{dt} - 2t^2 + 2x + 5\left(t\frac{dx}{dt} + 2x - t\right) = t$

[Putting the values of $2y$ from (3)]

or $t^2\frac{d^2x}{dt^2} + 8t\frac{dx}{dt} + 12x = 6t + 2t^2$

This is a Linear homogeneous D.E.

Putting $t = e^z$ we get

$\{D(D-1)+8D+12\}x = 6e^{z}+2e^{2z}$

$\therefore (D^2+7D+12)x = 6e^{z}+2e^{2z}$

A.E is $m^2+7m+12=0 \Rightarrow m=-3,-4$

$\therefore$ C.F is $c_1 e^{-3z}+c_2 e^{-4z}$

$$P.I = \frac{1}{D^2+7D+12}(6e^{z}+2e^{2z})$$

$$= 6 \cdot \frac{e^{z}}{1+7+12} + 2 \cdot \frac{e^{2z}}{4+14+12} = \frac{3}{10}e^{z}+\frac{1}{15}e^{2z}.$$

$$\therefore x = c_1 e^{-3z}+c_2 e^{-4z}+\frac{3}{10}e^{z}+\frac{1}{15}e^{2z}.$$

$$= \frac{c_1}{t^3}+\frac{c_2}{t^4}+\frac{3t}{10}+\frac{t^2}{15}. \quad \text{—— (I)}$$

Substituting this value in (3) we get

$$t\left\{-\frac{3c_1}{t^4}-\frac{4c_2}{t^5}+\frac{3}{10}+\frac{2t}{15}\right\}+\frac{2c_1}{t^3}+\frac{2c_2}{t^4}+\frac{3t}{5}$$
$$+\frac{2t^2}{15}-2y = t$$

$$\therefore y = -\frac{1}{2}c_1 t^{-3} - c_2 t^{-4}+\frac{2}{15}t^2-\frac{1}{20}t.$$

—— (II)

(I) & (II) together gives the required soln.

• Exercise -

① Solve $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$

$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t$

Ans $x = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + 2\cos t$

$y = c_1(\sqrt{2}-1)e^{\sqrt{2}t} + c_2(1-\sqrt{2})e^{-\sqrt{2}t} + 2\sin t$.

② Solve $t\,dx = (t-2x)\,dt$ ——— (1)

$t\,dy = (tx + ty + 2x - t)\,dt$ ——— (2)

Ans Adding (1) & (2) we get —

$t(dx + dy) = t(x+y)\,dt$

$\Rightarrow \frac{dx+dy}{x+y} = dt \quad \Rightarrow \frac{d(x+y)}{x+y} = dt$

Integrating we get $\log(x+y) = t + \log c_1$

$\Rightarrow \log\frac{(x+y)}{c_1} = t$

$\Rightarrow x+y = c_1 e^{t}$

$\Rightarrow y = c_1 e^{t} - x$ ——— (3)

From (1) we get $t\frac{dx}{dt} = t - 2x$

$\Rightarrow \frac{dx}{dt} + \frac{2}{t}x = 1$ — (*)

This is a 1st order linear D.E.

$I.F = e^{\int 2\frac{1}{t}dt} = e^{2\log t} = t^2$

Multiplying both sides of (*) by I.F and integrating we get

$$x t^2 = \int t^2 \, dt + c_2 = \frac{t^3}{3} + c_2$$

$$x = \frac{t}{3} + \frac{c_2}{t^2} \quad \text{———} \quad (4)$$

Using (4) in (3) we get

$$y = c_1 e^t - \frac{1}{3} t - c_2 t^{-2} \quad - (5)$$

(4) & (5) together gives the required Solⁿ.

③ solve $x \frac{dy}{dx} + z = 0$

$$x \frac{dz}{dx} + y = 0$$

Ans $y = c_1 x + c_2 \frac{1}{x}$

$z = -c_1 x + \frac{c_2}{x}$.

④ $\frac{dx}{dt} + 7x - y = 0$ Ans $x = e^{-6t}(c_1 \cos t + c_2 \sin t)$

$y = e^{-6t}[(c_1 + c_2)\cos t - (c_1 - c_2)\sin t]$.

$\frac{dy}{dt} + 2x + 5y$

⑤ $\frac{dx}{dt} + 2\frac{dy}{dt} - 2x + 2y = 3e^t$

$3\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 4e^{2t}$.

Ans $x = c_1 e^{-\frac{6t}{5}} + \frac{e^{2t}}{2} - \frac{3}{11} e^t$

$y = -8c_1 e^{-4t/5} + \frac{15}{22} e^t + c_2 e^{-t}$.

Reference: diff eqⁿ by Bej & Mukherjee or Maity & Ghosh.