VECTOR CALCULUS

Dot product or Scalar product :-

Let A and B be two vectors. Then Dot product or scalar product betn them is

$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta, \quad 0 \le \theta \le \pi$$

where θ is the angle betn the two vector.

(*) Suppose A, B and C are vectors and m is scalar. Then the following laws hold.

(i) $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$, commutative law for Dot product

(ii) $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$, Distributive law.

(iii) $m(\vec{A}\cdot\vec{B}) = (m\vec{A})\cdot\vec{B} = \vec{A}\cdot(m\vec{B}) = (\vec{A}\cdot\vec{B})m$

(iv) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ [where i, j, k be the unit vector corresponding to the rectangular system.

(v) If $\vec{A}\cdot\vec{B} = 0$ and A and B are not null vector, then A and B are perpendicular.

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$\therefore \vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3$.

Cross product :-

The cross products of vectors A and B is a vector $C = A \times B$ and θ be the angle of the two vector A and B.

The direction of $C = A \times B$ is perpendicular to the plane of A and B so that A, B and C form a right handed system.

$$\vec{A} \times \vec{B} = |A||B| \sin\theta \cdot \hat{n}$$ [$\hat{n}$ is the unit vector in the direction of $\vec{A} \times \vec{B}$]

If $\vec{A} = \vec{B}$ or if A is parallel to B, then $\sin\theta = 0$ and hence $\vec{A} \times \vec{B} = \vec{0}$.

(*) Suppose $\vec{A}, \vec{B}$ and $\vec{C}$ are vectors and m is a scalar, then the following laws holds.

(i) $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$. [commutative law doesn't holds]

(ii) $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$. [Distributive law]

(iii) $m(\vec{A} \times \vec{B}) = (m\vec{A}) \times \vec{B} = \vec{A} \times (m\vec{B}) = (\vec{A} \times \vec{B})m$

~~Saturday~~ (iv) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i}, \quad \hat{i} \times \hat{k} = -\hat{j}$.

(v) If $\vec{A} \times \vec{B} = 0$ and A and B are not null vector, then A and B parallel.

(*) Given $\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$

and $\vec{B} = B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}$

then $$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \hat{i}(A_2B_3 - A_3B_2) + \hat{j}(A_3B_1 - A_1B_3) + \hat{k}(A_1B_2 - A_2B_1)$$

Product of three vector (Triple product)

From three vector A, B and C we may obtain

1. A.(B×C), which is a scalar. This is known as Scalar Triple Product of A, B, C.
2. A×(B×C), which is a vector. This is known as vector Triple product

Note:- ~~A(B.C) is a vector A~~
A.(B.C) is a vector A multiplied by a scalar B.C.
A×(B.C) has no meanings.

Scalar Triple Product:-

Let A, B, C be three vectors with component (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_1, c_2, c_3) respectively with respect to the fundamental rectangular system $\hat{i}, \hat{j}, \hat{k}$.

Then we have,

$$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \hat{i}(b_2c_3 - b_3c_2) + \hat{j}(b_3c_1 - b_1c_3) + \hat{k}(b_1c_2 - b_2c_1)$$

and hence scalar triple product

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

The scalar triple product $\vec{A}\cdot(\vec{B}\times\vec{C})$ is also written as $[\vec{A}\ \vec{B}\ \vec{C}]$

<u>Note:-</u> If a, b, c be three vectors, then

1. $b\cdot(c\times a) = (c\times a)\cdot b = -b\cdot(a\times c) = -(a\times c)\cdot b$
   $c\cdot(a\times b) = (a\times b)\cdot c = -c\,(b\times a) = -(b\times a)\cdot c$
   $a\cdot(b\times c) = (b\times c)\cdot a = -a\,(c\times b) = -(c\times b)\cdot a$

   In box notation,
   $[a\,b\,c] = [b\,c\,a] = [c\,a\,b] = -[b\,a\,c]$
   $= -[a\,c\,b]$ etc.

2. If three non-zero vectors a, b, c are co-planar, then their scalar triple product is zero i.e. $[a\,b\,c] = 0$.

3. The scalar triple product $[a\,b\,c]$ is numerically equal to the volume V of a parallelopiped having a, b, c as coterminous edges.

4. The relation of the vectors of the fundamental system (i-j-k system) as
   $$[\hat{i}\,\hat{j}\,\hat{k}] = [\hat{j}\,\hat{k}\,\hat{i}] = [\hat{k}\,\hat{i}\,\hat{j}] = 1$$
   and $[\hat{i}\,\hat{k}\,\hat{j}] = [\hat{k}\,\hat{j}\,\hat{i}] = [\hat{j}\,\hat{i}\,\hat{k}] = -1.$

Vector triple product:-

We now consider the cross product of a and $b \times c$ i.e., $a \times (b \times c)$

$$a \times (b \times c) = (a \cdot c) b - (a \cdot b) c.$$

Corollary:-

1. $b \times (c \times a) = (b \cdot a) c - (b \cdot c) a = (a \cdot b) c - (b \cdot c) a$
2. $(a \times b) \times c = -c \times (a \times b) = -[(c \cdot b) a - (c \cdot a) b]$
   $= (c \cdot a) b - (c \cdot b) a.$
3. $a \times (b \times c)$, $b \times (c \times a)$ and $c \times (a \times b)$ lies on the same plane.

Example:-

Q. Let $a = -3\hat{i} + 7\hat{j} + 5\hat{k}$, $b = -5\hat{i} + 7\hat{j} - 3\hat{k}$, $c = 7\hat{i} - 5\hat{j} - 3\hat{k}$. Find $a \times (b \times c)$ in terms of $\hat{i}, \hat{j}, \hat{k}$ and then verify $a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$.

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Sol:-

$$b \times c = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix} = -36\hat{i} - 36\hat{j} - 24\hat{k}$$

$$a \times (b \times c) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 7 & 5 \\ -36 & -36 & -24 \end{vmatrix} = 12\hat{i} - 252\hat{j} + 360\hat{k}.$$

Again $(a \cdot c) = -71$, $(a \cdot b) = 49$

$$(a \cdot c) b - (a \cdot b) c = -71(-5\hat{i} + 7\hat{j} - 3\hat{k}) - 49(7\hat{i} - 5\hat{j} - 3\hat{k})$$
$$= 12\hat{i} - 252\hat{j} + 360\hat{k}.$$

Thus we have

$$a \times (b \times c) = (a \cdot c) b - (a \cdot b) c = 12\hat{i} - 252\hat{j} + 360\hat{k}.$$

Product of four vector :-

A. Scalar product of four vectors.

We consider the scalar product of $(a \times b)$ and $(c \times d)$ i.e., we obtain

$$(a \times b) \cdot (c \times d)$$

Interchanging dot and cross we may write

$$a \cdot b \times (c \times d) = a \cdot [(b \cdot d)c - (b \cdot c)d]$$
$$= (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$

We may also write this result in the form

$$\begin{vmatrix} a \cdot c & a \cdot d \\ b \cdot c & b \cdot d \end{vmatrix}$$

This form is called Lagrange's Identity.

B. Vector product of four vectors.

We consider the vector product of $(a \times b)$ and $(c \times d)$ as $(a \times b) \times (c \times d)$ which is a vector.

$$(a \times b) \times (c \times d) = [a\,b\,d]c - [a\,b\,c]d$$
$$= [c\,d\,a]b - [c\,d\,b]a$$

① If any vector d can be written as a linear combination of three non-coplanar vectors of a, b, c (i.e., $[a\,b\,c] \neq 0$), then $d = xa + yb + zc$

where $x = \frac{[d\,b\,c]}{[a\,b\,c]}$, $y = \frac{[d\,c\,a]}{[a\,b\,c]}$, $z = \frac{[d\,a\,b]}{[a, b\,c]}$

Thus, $d = \frac{[d\,b\,c]a}{[a\,b\,c]} + \frac{[d\,c\,a]b}{[a\,b\,c]} + \frac{[d\,a\,b]c}{[a\,b\,c]}$