Name - Bishnupriya Jana
Department - physics
Sem - II (H)
Paper - C3T

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Topic - Ampere's circuital law and magnetic force

Ampere's circuital law: = The line integral of the magnetic field around any closed path is equal to μ_0 times the total current enclosed by the path

i.e $\oint \vec{B}.\vec{dl} = \mu_0 i$

Differential form: =

Ampere's circuital law,

We have,

$$\oint \vec{B}.\vec{dl} = \mu_0 i$$

Again $i = \int \vec{J}.\hat{n}\,ds$

where $\vec{J}$ is the current density vector.

or, $\oint \vec{B}.\vec{dl} = \mu_0 \int_s \vec{J}.\hat{n}\,ds$

According to stoke's theorem

$$\int_s \vec{\nabla}\times\vec{B}.\hat{n}\,ds = \int_s \mu_0 \vec{J}.\hat{n}\,ds$$

or, $\int_s (\vec{\nabla}\times\vec{B} - \mu_0\vec{J}).\hat{n}\,ds = 0$

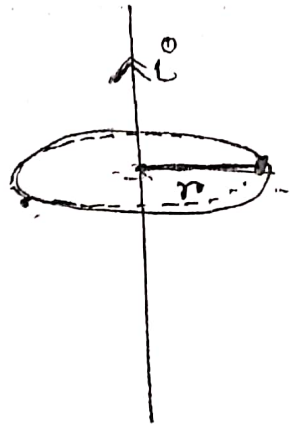
$$\boxed{\therefore \vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$$

This is the differential equation of Ampere's circuital law.

Applications of Ampere's circuital law :

Magnetic field due to a straight long current carrying conduc

Consider a straight long conductor carries a current i. We want to find the magnetic field at any point P at a distance r from the axis of the conductor.



We draw a circle of radius r through the point P with centre on the axis of the conductor, called Amperian loop.

From Ampere's circuital law, $\oint \vec{B}\cdot\vec{dl} = \mu_0 i$

or, $B\oint dl = \mu_0 i$

or, $B \times 2\pi r = \mu_0 i$

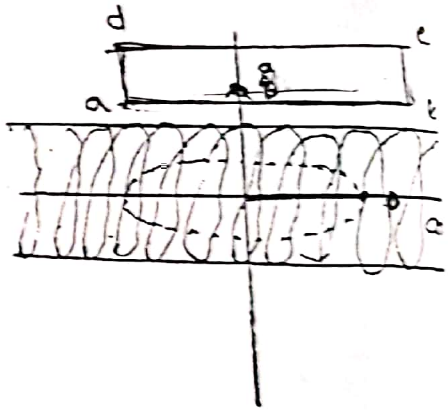
or, $\boxed{B = \frac{\mu_0 i}{2\pi r}}$

The direction of $\vec{B}$ circles around the condu... according to right hand cork screw rule.

② Magnetic field due to a long solenoid :

Let us consider a long solenoid consisting of n turns per un... length closely wound on a circular cylinder of radius a and carri... a steady current i.

(i) First we want to find magnetic field outside the solenoid $(r>a)$



Consider an Amperian loop in the form of a rectangle abcd which lies entirely outside the solenoid. Let the distances of the sides from the axis of the solenoid are r_1 and r_2.

From Ampere's circuital law,

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 i = 0$$

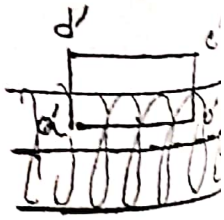
$$\text{or, } [B(r_2) - B(r_1)]\,l = 0$$

$$\text{or, } B(r_2) = B(r_1)$$ which indicates that field outside does not depend on distance from the axis.

Again $B \to 0$ as $r \to \infty$ which indicates that the field outside must be zero everywhere.

$\therefore$ $\boxed{B=0}$ when $r>a$

(ii). Magnetic field inside the solenoid $(r<a)$:



Consider another Amperian loop a'b'c'd' rectangular in shape which is half inside and half outside.

From Ampere's circuital law,

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 (ni) l$$

or, $\int_{a'b'} \vec{B}\cdot\vec{dl} + \int_{b'c'} \vec{B}\cdot\vec{dl} + \int_{c'd'} \vec{B}\cdot\vec{dl} + \int_{d'a'} \vec{B}\cdot\vec{dl} = \mu_0 n i l$

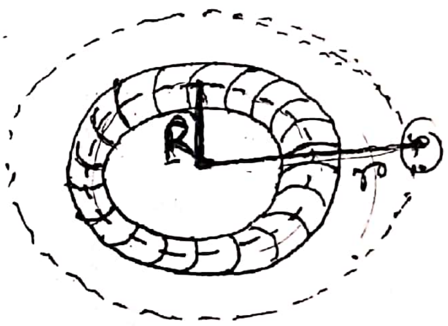
or, $B\cdot l + 0 + 0 + 0 = \mu_0 n i l$.

or, $\boxed{B = \mu_0 n i}$ which is directed along the axis of the solenoid.

$r < a$

(3) Magnetic field due to a toroid :-

A long wire wound closely around a circular ring forms a toroid. The lines of force would be circles whose centres lie on the axis of the toroid passing through O.



To apply Ampere's law, we consider one such circle as an Amperian loop.

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 N i$$

or $B \oint dl = \mu_0 N i$

or $B \times 2\pi R = \mu_0 N i$

or, $B = \dfrac{\mu_0 N i}{2\pi R}$ where N is the total no. of turns

$\dfrac{N}{2\pi R} = n$

$\therefore \boxed{B = \mu_0 n i}$

$n = \dfrac{N}{2\pi R}$

Outside: $B = 0$

Motion of a charged particle in a uniform magnetic field :

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Suppose a charged particle of charge q and mass m moves with a velocity $\vec{v}$ in a region of uniform magnetic field $\vec{B}$ where $\vec{B} = B\hat{k}$. Equation of motion of the particle can be written as,

$$m\frac{d\vec{v}}{dt} = q(\vec{v}\times\vec{B})$$

or $$m\left(\hat{i}\frac{dv_x}{dt} + \hat{j}\frac{dv_y}{dt} + \hat{k}\frac{dv_z}{dt}\right) = q\left(\hat{i}Bv_y - \hat{j}Bv_x\right)$$

Therefore, $$m\frac{dv_x}{dt} = qBv_y \quad \text{——— (1)}$$

$$m\frac{dv_y}{dt} = -qBv_x \quad \text{——— (2)}$$

$$m\frac{dv_z}{dt} = 0 \quad \text{——— (3)}$$

- Equation (3) shows that, v_z = Constant and parallel to $\vec{B}$ $= v_{\parallel}$ (say).

or $$\frac{dz}{dt} = v_{\parallel}$$

or $$z = v_{\parallel}t + z_0 \quad \text{——— (4)}$$

- Multiplying equation (2) by $j\,(=\sqrt{-1})$ and adding with equation (1) we get,

$$\frac{d}{dt}(v_x + jv_y) = \frac{qB}{m}(v_y - jv_x)$$

or, $$\frac{d}{dt}(v_x + jv_y) = -j\omega(v_x + jv_y) \quad \text{——— (5)}$$

Where $\omega = \frac{qB}{m}$ is called cyclotron frequency.

Solution of equation (5) is,

$$v_x + jv_y = Ce^{-j\omega t} = C\cos\omega t - jC\sin\omega t$$

$\therefore$ $v_x = C\cos\omega t$ and $v_y = -C\sin\omega t$

or $\frac{dx}{dt} = C\cos\omega t$ or, $\frac{dy}{dt} = -C\sin\omega t$

or, $x = \frac{C}{\omega}\sin\omega t + x_0$ or, $y = \frac{C}{\omega}\cos\omega t + y_0$

or $x - x_0 = A\sin\omega t$ ——— (6) or, $y - y_0 = A\cos\omega t$ ——— (7)

Combining equation (6) and (7), we get,

$$\boxed{(x-x_0)^2 + (y-y_0)^2 = A^2} \quad \text{--- (8)}$$

Equation (8) represents a circle in the x-y plane with the centre at (x_0, y_0).

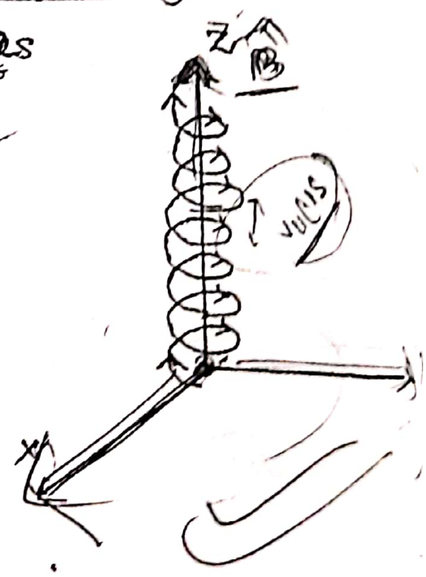
Note: (i) If $v_{\parallel} = 0$, then the particle describes a circular path in the plane perpendicular to $\vec{B}$. Let the velocity of the particle is $v_{\perp}$.

$$v_{\perp}^2 = \dot{x}^2 + \dot{y}^2 = A^2\omega^2 \quad \text{as} \quad \dot{x} = \omega A \cos\omega t$$
$$\dot{y} = -\omega A \sin\omega t$$

or, $A = \dfrac{v_{\perp}}{\omega}$

or $\boxed{A = \dfrac{v_{\perp} m}{qB}}$ Radius of the circular path, called radius of gyration.

(ii) If $v_{\parallel} \neq 0$, then the particle describes a helical path with $\vec{B}$ as axis.



Motion of a charged particle in crossed electric and magnetic field:

Suppose a charged particle of charge q and mass m moves with a velocity $\vec{v}$ in a region of crossed electric field ($\vec{E}$) and magnetic field ($\vec{B}$) where $\vec{B}$ is along z-axis and $\vec{E}$ is along y-axis.

Therefore the equation of motion of the particle,

$$m\frac{d\vec{v}}{dt} = q(\vec{E}+\vec{v}\times\vec{B}) \qquad \text{Now } \vec{v}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix}$$

or $$m\frac{d\vec{v}}{dt} = q\left[E\hat{j} + (\hat{i}Bv_y - \hat{j}Bv_x)\right]$$

or, $$\hat{i}\frac{dv_x}{dt} + \hat{j}\frac{dv_y}{dt} + \hat{k}\frac{dv_z}{dt} = \hat{i}\frac{qB}{m}v_y + \hat{j}\left(\frac{qE}{m} - \frac{qB}{m}v_x\right).$$

Whose component equations are,

$$\frac{dv_x}{dt} = \omega v_y \quad \text{—— (1)}$$

$$\frac{dv_y}{dt} = a - \omega v_x \quad \text{—— (2)}$$

$$\frac{dv_z}{dt} = 0 \quad \text{—— (3)}$$

Where, $\left\{ \frac{qB}{m} = \omega, \quad \frac{qE}{m} = a \right.$

We find the solutions assuming the charged particle is at rest at the origin ($x=0, y=0, z=0$) at $t=0$.

From equation (3), we get $v_z = \text{Const}$ (say $v_{\parallel}$)

or $\boxed{z=0}$ always.

Differentiating equation (2) w.r.to t,

$$\frac{d^2v_y}{dt^2} = -\omega\frac{dv_x}{dt}$$

$$= -\omega(\omega v_y) \quad \text{using equation (1).}$$

or, $$\frac{d^2v_y}{dt^2} = -\omega^2 v_y \quad \text{—— (4)}$$

Solution of equation (4), gives,

$$v_y = A\sin\omega t + B\cos\omega t \quad \text{—— (5)}$$

At $t=0$, $v_y=0$ Therefore $B=0$

Thus $v_y = A\sin\omega t$ —— (5)

Integrating, $y = -\frac{A}{\omega}\cos\omega t + c_1$

Again at $t=0$, $y=0$, Therefore $c_1 = \frac{A}{\omega}$

$\therefore$ ~~$y = \frac{A}{\omega}\cos\omega t$ —— (6)~~

$$\therefore \boxed{y = \frac{A}{\omega}(1-\cos\omega t)} \quad \text{—— (6)}$$

Substituting the value of v_y (eqn 5) in equation (1), we get-

$$\frac{dv_x}{dt} = \omega A\sin\omega t$$

Integrating, $v_x = -A\cos\omega t + c_2$

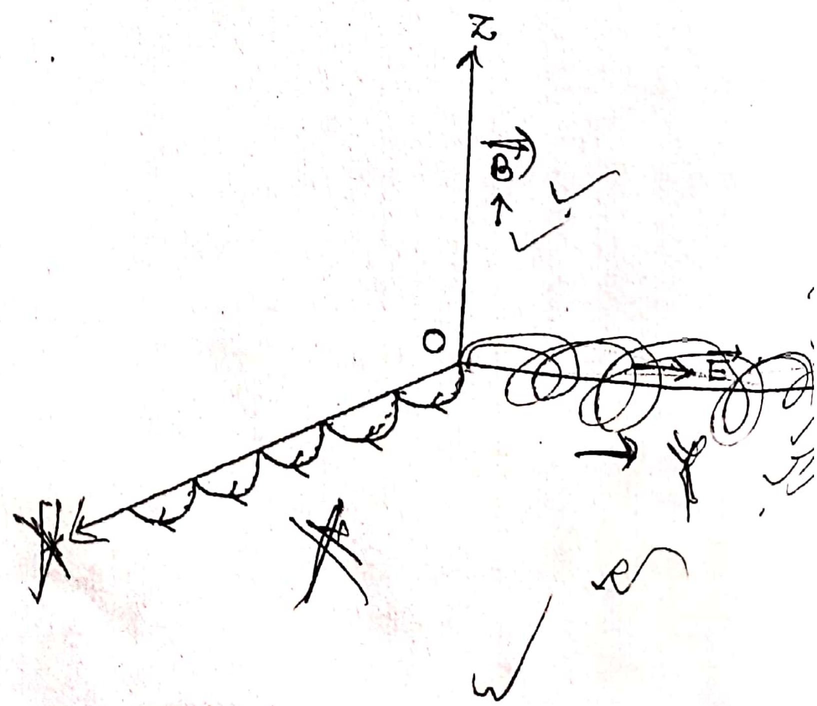
At $t=0$, $v_x=0$ Thus $c_2 = A$

$$\therefore v_x = A(1-\cos\omega t) \quad \text{—— (7)}$$

Integrating, $\boxed{x = \frac{A}{\omega}(\omega t - \sin\omega t)}$ —— (8) as: at $t=0$, $\therefore$ const.

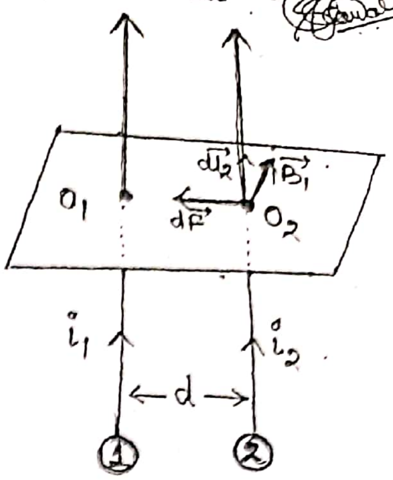
Equation (6) and (8) are the parametric equations of a cyclo
This is the path generated by a point on the rim of a wheel rolling along a line.

equation of cycloid



Force between two parallel current carrying conductors :

Suppose two long parallel conductors carrying current i_1 and i_2 are separated by a distance d.



The conductor 1 produces a mag. field at the position of conductor 2 (O_2).

$\therefore \vec{B_1} = \frac{\mu_0 i_1}{2\pi d}$ directed perpendicular to the conductor 2.

Force acting on an element $\vec{dl_2}$ of conductor 2 due to $\vec{B_1}$,

$$d\vec{F} = i_2 \vec{dl_2} \times \vec{B_1}$$

$$= \frac{\mu_0 i_1 i_2}{2\pi d} dl_2$$ which is directed towards conductor 1.

Therefore force per unit length on conductor 2 due to $\vec{B_1}$

$$\vec{F} = \frac{\mu_0 i_1 i_2}{2\pi d}$$ which is directed towards conductor 1.

Similarly force per unit length on conductor 1 due to $\vec{B_2}$

$$\vec{F} = \frac{\mu_0 i_1 i_2}{2\pi d}$$ which is directed towards conductor 2.

i.e force is attractive if i_1 and i_2 flows in same direction.

Note: If i_1 and i_2 flows in opposite direction, $\vec{F}$ is same as like above but in opposite direction i.e away from each other i.e the force is repulsive.

Remember that: Magnitude of mutual force per unit length

$$\boxed{|\vec{F}| = \frac{\mu_0 i_1 i_2}{2\pi d}}$$

Magnetic scalar potential :

Ampere's law for steady current i.e $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ indicates that $\vec{B}$ is not a conservative field.

If the current density is zero in some region of space then $\vec{\nabla}\times\vec{B}=0$ i.e $\vec{B}$ can be described as a conservative field. In such region $\vec{B}$ can be expressed as the gradient of a scalar potential,

i.e $\boxed{\vec{B} = -\mu_0\vec{\nabla}\phi}$ where ϕ is called the magnetic scalar potential whose unit is "ampere".

Now, $\vec{\nabla}\cdot\vec{B} = 0$

or, $\vec{\nabla}\cdot(-\mu_0\vec{\nabla}\phi) = 0$

or, $\boxed{\nabla^2\phi = 0}$ i.e magnetic scalar potential satisfies Laplace's equation.

Magnetic scalar potential of a small current loop :-

Magnetic vector potential due to a current loop,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\cdot\frac{\vec{m}\times\vec{R}}{R^3}$$ where $\vec{m}$ is the magnetic moment.

$$\therefore \vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\cdot\vec{\nabla}\times\left(\frac{\vec{m}\times\vec{R}}{R^3}\right)$$

$$= \frac{\mu_0}{4\pi}\left[\frac{1}{R^3}\vec{\nabla}\times(\vec{m}\times\vec{R}) + \vec{\nabla}\left(\frac{1}{R^3}\right)\times(\vec{m}\times\vec{R})\right]$$

$$= \frac{\mu_0}{4\pi}\left[\frac{1}{R^3}\cdot 2\vec{m} + \left(-\frac{3\vec{R}}{R^5}\right)\times(\vec{m}\times\vec{R})\right]$$

$$= \frac{\mu_0}{4\pi}\left[\frac{2\vec{m}}{R^3} - \frac{3}{R^5}(\vec{R}\times\vec{m}\times\vec{R})\right]$$

$$= \frac{\mu_0}{4\pi}\left[\frac{2\vec{m}}{R^3} - \frac{3}{R^5}\left\{\vec{m}R^2 - \vec{R}(\vec{R}\cdot\vec{m})\right\}\right]$$

$$= \frac{\mu_0}{4\pi}\left[-\frac{\vec{m}}{R^3} + \frac{3(\vec{m}\cdot\vec{R})\vec{R}}{R^5}\right]$$

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\vec{R})\vec{R}}{R^5} - \frac{\vec{m}}{R^3}\right]} = -\mu_0\vec{\nabla}\left(\frac{\vec{m}\cdot\vec{R}}{4\pi R^3}\right)$$

The magnetic field due to a small current loop at a point P is given by,

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\vec{R})\vec{R}}{R^5} - \frac{\vec{m}}{R^3}\right]$$

$$= -\mu_0 \vec{\nabla}\left(\frac{\vec{m}\cdot\vec{R}}{4\pi R^3}\right) \text{ ——— (1)}$$

If ϕ be the magnetic scalar potential,

$$\vec{B} = -\mu_0 \vec{\nabla}\phi \text{ ——— (2)}$$

Compairing equation (1) and (2), we may write,

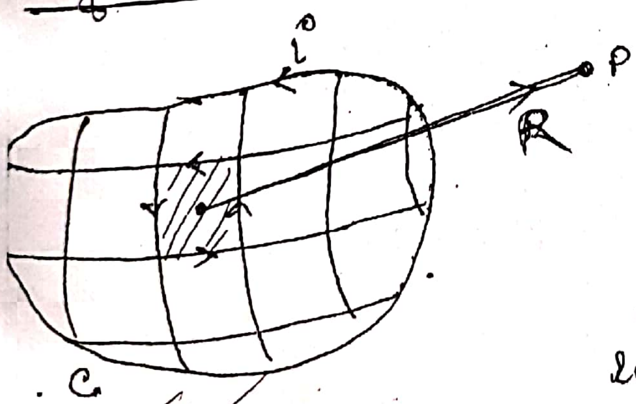
$$\boxed{\phi = \frac{\vec{m}\cdot\vec{R}}{4\pi R^3}}$$

Which is magnetic scalar potential due to a small current loop. where $\vec{m}$ is the magnetic moment associated with the current loop.

Remember that: magnetic moment = product of current and area

i.e $\vec{m} = i\vec{A}$

We extend the above to find magnetic scalar potential due to a large current loop of arbitrary shape:



The large current loop C may be divided into a number of small loops around which there flows the same current i.

The magnetic moment of the elementary loop, $d\vec{m} = i\,d\vec{s}$

Scalar potential at P due to the elementary loop,

$$d\phi = \frac{d\vec{m}\cdot\vec{R}}{4\pi R^3}$$

Total magnetic scalar potential at P,

$$\phi = \frac{1}{4\pi}\int \frac{d\vec{m}\cdot\vec{R}}{R^3} = \frac{i}{4\pi}\int \frac{d\vec{s}\cdot\vec{R}}{R^3}$$

or $\phi = \frac{i}{4\pi}\int d\Omega$ where $d\Omega = \frac{\vec{dS}\cdot\vec{R}}{R^3}$, solid angle subtended by the ~~loop~~ elementary loop at P.

$\int d\Omega = \Omega$, solid angle subtended by the loop C at P.

$$\therefore \boxed{\phi = \frac{i}{4\pi}\Omega}$$

Magnetic shell :

- A thin sheet of magnetic material, which is magnetised everywhere at right angles to its surface is called a magnetic shell.

- The ~~streg~~ strength of the magnetic shell is defined as the magnetic moment per unit facial area.

i.e $\vec{\tau} = \frac{\text{magnetic moment}}{\text{facial area}} = \frac{m}{A} = \frac{iA}{A} = i$

- Magnetisation :- Magnetic dipole moment per unit volume of a magnetic material is called magnetisation.

- Another defination of the strength of a magnetic shell as the product of the magnetisation ($\vec{M}$) and the thickness of the shell (t).

i.e $\boxed{\vec{\tau} = \vec{M}\cdot t}$

$$\vec{M} = \lim_{\Delta v \to 0} \frac{\Delta M}{\Delta v}$$

Ampere's equivalence theorem :

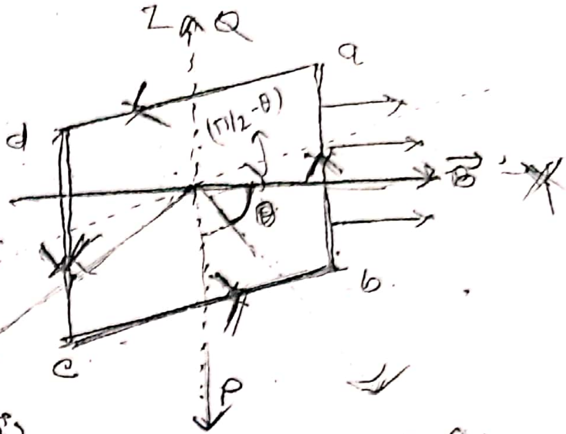
The strength of a magnetic shell (due to a current loop) $= \frac{\text{magnetic moment}}{\text{area}}$

$= \frac{\text{current} \times \text{area}}{\text{area}} = \text{current}$

i.e the strength of a equivalent magnetic shell is equal to the current in the loop.

"A current carrying loop produces the same magnetic field as is produced by a magnetic shell whose boundary coincides with the boundary of the loop and whose strength is equal to the current flowing in the loop." — Theorem.

Torque, Potential energy and Translational force due to a current loop in an external magnetic field.



Consider a rectangular current loop abcd carrying current i placed in a uniform magnet field $\vec{B}$.

Let $ab = cd = l_1$ along Y axis, making angle
$bc = da = l_2$ $(\frac{\pi}{2}-\theta)$ with Y axis.

$\therefore$ Area $A = l_1 l_2$

Force acting on side ab, $\vec{F_1} = i(\vec{l_1}\times\vec{B})$
$= i l_1 B\,\hat{n}$ —— (1) $\hat{n} = (-\hat{i})$, $\theta = 90°$

Force acting on side cd, $\vec{F_2} = i(\vec{-l_1}\times\vec{B})$
$= i l_1 B\,(+\hat{n})$ —— (2) $\hat{n} = \hat{i}$

Since, Again force acting on sides bc and da is ~~zero (as $\vec{l} \parallel \vec{B}$)~~ same in magnitude but opposit in direction and acting along same line. They cancel one another.

The loop is free to rotate about the axis PQ due to equal and opposite forces on ab and cd.

i.e. Torque acting on the loop $\vec{\tau} = \vec{F_1}\times\vec{l_2} = \vec{l_2}\times\vec{F_1}$
$= F_1 \cdot l_2 \sin\theta\,\hat{n_0}$; $\boxed{\theta = (\frac{\pi}{2}+\frac{\pi}{2}-\theta)} = \pi - \theta$
$= i l_1 l_2 B \sin\theta\,\hat{n_0}$
$= i A B \sin\theta\,\hat{n_0}$
$= i\vec{A}\times\vec{B}$

$\vec{F}_{cb} = iBl_1\cos\theta\,(-\hat{k})$

$\vec{F}_{ad} = iBl_1\sin(\pi/2-\theta)(\hat{k})$

$\therefore \boxed{\vec{\tau} = \vec{m}\times\vec{B}}$ where $\vec{m} = i\vec{A}$, is the dipole moment of the loop.

Total torque
$\vec{\tau} = N i(\vec{A}\times\vec{B})$

$\tau = mB\sin\theta$; torque tends to decrease the angle θ

If U be the potential energy of the loop, then
$$\tau = \frac{\partial U}{\partial \theta} = mB\sin\theta$$

Integrating, $U = -mB\cos\theta + C$ At $\theta = \pi/2$, $U = 0$
$\therefore C = 0$

~~Ex~~ Mechanism of a electric fan.

$\therefore\ U = -mB\cos\theta$

$$\boxed{U = -\vec{m}\cdot\vec{B}}$$

A magnetic dipole of moment $\vec{m}$ is placed in a magnetic field $\vec{B}$, then it experiences a translational force $\vec{F}$,

~~then~~ $\vec{F} = -\vec{\nabla}U = -\vec{\nabla}(-\vec{m}\cdot\vec{B})$

or, $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$

$= (\vec{B}\cdot\vec{\nabla})\vec{m} + (\vec{m}\cdot\vec{\nabla})\vec{B} + \vec{B}\times(\vec{\nabla}\times\vec{m}) + \vec{m}\times(\vec{\nabla}\times\vec{B})$

Since $\vec{m}$ does not involve space co-ordinates,

$(\vec{B}\cdot\vec{\nabla})\vec{m} = 0$ and $\vec{\nabla}\times\vec{m} = 0$

$\vec{m} = \vec{m}(r_0)$

$\vec{\nabla} \{ \vec{\nabla}(r) \Rightarrow$ field point

$= (\vec{m}\cdot\vec{\nabla})\vec{B} + \vec{m}\times\cancel{(\vec{\nabla}\times\vec{B})}^{0}$ as $\vec{\nabla}\times\vec{B} = 0$

$$\boxed{\vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B}}$$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

Here $J=0$

$\Rightarrow \vec{\nabla}\times\vec{B} = 0$

if $\vec{B}$ is uniform

then $\vec{F} = \vec{0}$

translational - force

~~for~~ electrostatics, $\vec{F_e} = (\vec{p}\cdot\vec{\nabla})\vec{E}$

Problem base questions :=

1. Starting from Bio-Savart law establish:

   (a) Ampere's circuital law;

   (b) the relation $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

2. Find the magnetic vector potential $\vec{A}$ at ~~using~~ a distance r from an infinite straight wire carrying current I. Hence calculate the magnetic field.

3. Apply the Ampere's circuital law to find the magnetic field due to;

   (a) a long straight current carrying conductor,

   (b) a long current carrying solenoid at an internal and external points,

   (c) a current carrying toroid at an internal and external points,

   (d) and infinite sheet of current.