12.10 Stefan-Boltzmann law : Total energy of radiation

The dependence of total radiation from a radiator on its temperature was first studied experimentally by Stefan. He concluded that the *total energy of radiation emitted by a black body is proportional to the fourth power of its absolute temperature*. On the supposition that the black-body radiation behaves as a perfect gas, Boltzmann applied the laws of thermodynamics to radiation and deduced the Stefan law *theoretically*. Since then, the law is generally referred to as *Stefan-Boltzmann law*. It states:

If a black body at an absolute temperature T be surrounded by another black body at absolute temperature T_1, the amount of radiation emitted per sec per unit area of the former is

$$Q = \sigma(T^4 - T_1^4) \tag{12.10.1}$$

where σ is known as *Stefan constant*.

Derivation — Let u be the energy density of radiation in an enclosure at constant absolute temperature T, V be the volume of the enclosure and p the pressure of radiation. So the total energy of radiation in the enclosure is $U = uV$ and p and u are functions of T.

From the first law of thermodynamics,

$$\begin{aligned} dQ &= dU + pdV = d(uV) + \tfrac{1}{3}udV \quad (\because \quad p = u/3) \\ &= udV + Vdu + \tfrac{1}{3}udV \\ &= Vdu + \tfrac{4}{3}udV \end{aligned} \tag{12.10.2}$$

From the second law of thermodynamics,

$$dQ = TdS \tag{12.10.3}$$

Using (12.10.2) and (12.10.3), $$dS = \frac{V du}{T} + \frac{4}{3}\frac{u}{T}dV \tag{12.10.4}$$

But, since

$$S = f(u, V)$$

$$dS = \left(\frac{\partial S}{\partial u}\right)_V du + \left(\frac{\partial S}{\partial V}\right)_u dV \tag{12.10.5}$$

From (12.10.4) and (12.10.5), we thus have

$$\left(\frac{\partial S}{\partial u}\right)_V = \frac{V}{T}; \qquad \left(\frac{\partial S}{\partial V}\right)_u = \frac{4}{3}\frac{u}{T} \tag{12.10.6}$$

Again, since dS is a perfect differential,

$$\frac{\partial^2 S}{\partial V \partial u} = \frac{\partial^2 S}{\partial u \partial V}$$

$\therefore$ From (12.10.6) $$\frac{\partial}{\partial V}\left(\frac{V}{T}\right) = \frac{4}{3}\frac{\partial}{\partial u}\left(\frac{u}{T}\right) \tag{12.10.7}$$

Now, the temperature T is independent of V and is a function of u alone. So, from (12.10.7)

$$\frac{1}{T} = \frac{4}{3}\left(\frac{1}{T} - \frac{u}{T^2}\frac{dT}{du}\right)$$

$$\Rightarrow \quad 1 = \frac{4}{3} - \frac{4}{3}\frac{u}{T}.\frac{dT}{du}$$

or, $$4\frac{dT}{T} = \frac{du}{u}$$

Integrating, we obtain $$\ln u = 4 \ln T + \ln \alpha \text{ (const.)}$$

$$\therefore \quad u = \alpha T^4 \tag{12.10.8}$$

where α is called the *total density radiation constant*.

As the radiation lost per second per unit area of the enclosure, i.e. Q, is proportional to the energy density u, Q is also proportional to T^4. Hence

$$Q = \sigma T^4 \tag{12.10.9}$$

This is **Stefan-Boltzmann law,** where σ is the *Stefan constant*.

When this body is surrounded by another at temperature T_1, the latter will also emit radiation equal to σT_1^4 which will be incident on to the first body. So the net loss of radiation of the first body per sec per unit area would be

$$Q = \sigma(T^4 - T_1^4) \tag{12.10.10}$$

which is the same as given in (12.10.1).

It can be shown *theoretically* ('Note' below) that the Stefan constant σ is given by

$$\sigma = \tfrac{1}{4}\alpha c \tag{12.10.11}$$

where c is the velocity of light.

[**Note. Deduction of** (12.10.11) : Let an area dA be placed in space containing radiation. The amount of radiation crossing dA in a direction making an angle θ with the normal to dA is proportional to $dA \cos\theta$. Consider the radiation that propagates through any point in dA from below above. Again, the amount of radiation confined to a small solid angle is proportional to the solid angle.

$\therefore$ The amount of radiation propagated through dA in a direction defined by θ and $\theta + d\theta$, and ϕ and $\phi + d\phi$ in time dt is

$$dQ = KdA\cos\theta.\sin\theta d\theta d\phi.dt$$

$\therefore$ Total radiation through dA for all possible directions is given by

$$Q = KdAdt\int_{\circ}^{\pi/2}\sin\theta\cos\theta d\theta\int_{\circ}^{2\pi}d\phi$$

$$= \tfrac{1}{2}KdAdt.2\pi = \pi KdAdt$$

$\therefore$ Total radiation passing through unit area in space in unit time is :

$$Q = \pi K = \tfrac{1}{4}cu$$

$(\because$ energy density of radiation, $u = 4\pi K/c)$

Now since, $u = \alpha T^4$, $Q = \frac{1}{4}\alpha c T^4 = \sigma T^2$. Therefore, $\sigma =$ Stefan constant $= \frac{1}{4}\alpha c]$

12.10.1 Newton's law of cooling from Stefan's law

Newton's law of cooling states that *the rate at which a body loses heat due to radiation is directly proportional to the excess temperature, that is, the temperature-difference between the body and the surroundings, provided this difference is small.* The law follows easily from Stefan-Boltzmann law.

Let $T_\circ$ and T_1 be the absolute temperatures of the body and the surroundings. Then, by Stefan's law, the rate of loss of heat per unit area of the body is

$$Q = \sigma(T_\circ^4 - T_1^4) = \sigma(T_\circ^2 - T_1^2)(T_\circ^2 + T_1^2)$$

$$= \sigma(T_\circ - T_1)(T_\circ + T_1)(T_\circ^2 + T_1^2)$$

$$= \sigma(T_\circ - T_1)(T_\circ^3 + T_\circ^2 T_1 + T_\circ T_1^2 + T_1^3) \quad (12.10.12)$$

If the excess temperature $(T_\circ - T_1)$ be small, $T_\circ \simeq T_1$. So, from equation (12.10.12)

$$Q \simeq \sigma(T_\circ - T_1) \times 4T_1^3 = \beta(T_\circ - T_1)$$

where $\beta = 4T_1^3\sigma$, a constant, since T_1 is constant.

$$\therefore \quad Q \propto (T_\circ - T_1)$$

This is the familiar *Newton's law of cooling.*

12.10.2 Experimental verification of Stefan's law

Stefan's law was experimentally verified by Lummer and Pringsheim using total radiation from a black body over a wide range of temperatures from 100°C to about 1300°C.

Arrangement — It consists essentially of a *hollow copper sphere C* (Fig.12.11) painted inside with *platinum black* and kept in a *bath of fused nitre.* The copper sphere could be heated up to 600°C in this bath which is well stirred. It thus serves as a black body. For a temperature range between 900 – 1300°C, the copper sphere is replaced by an iron cylinder coated inside with *platinum black* and the bath by a *gas furnace.* The temperature of the bath is measured by a *thermocouple Th.* The flow of radiation is controlled by the *water-cooled shutter* S_1 in front of the *opening O* and the quantity of radiation coming out of O is measured by the bolometer B, standardised with the help of another blackbody A, coated inside with *platinum black* and maintained at a constant temperature of 100°C.

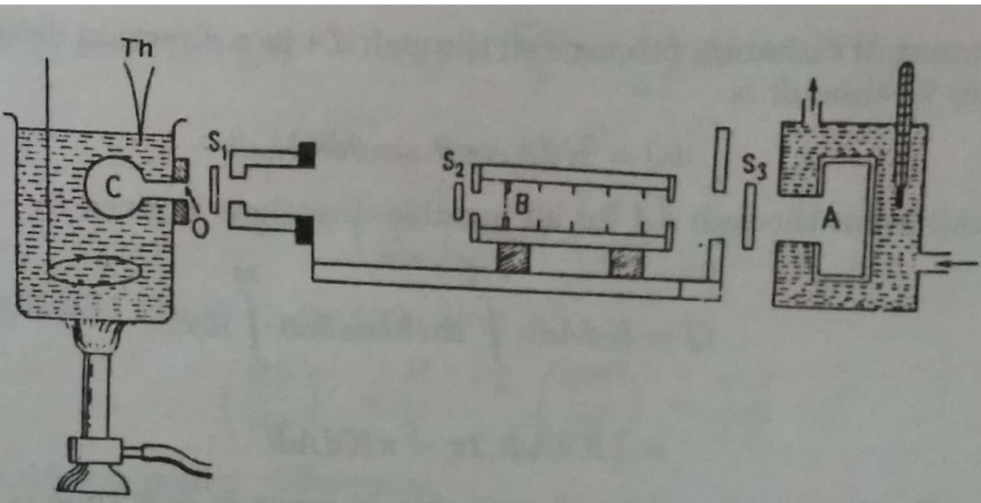

Fig.12.11. Verification of Stefan's law

The bath is heated to the desired steady temperature and the shutter raised to enable radiations fall on the receiving face of the bolometer. When the galvanometer (not shown) shows a steady deflection, it is noted. Observations are taken at different temperatures of the bath.

The above readings are next expressed in terms of deflection produced when A, maintained at 100°C, is kept at a standard distance of 633 mm. The deflection d of the galvanometer for the black body C at a temperature T K is found to vary as

$$d = \alpha\,(T^4 - 290^4)$$

where 290K (=17°C) is the temperature of the shutter S_1.

From the data of Lummer and Pringsheim at different T's, α was found to be a constant and this verified the Stefan's law.

Note. The constant α however is not the Stefan constant but a constant that depends on the calibration of the bolometer.

12.11 Energy distribution in black body radiation : Wien's law

Stefan-Boltzmann law shows how the total energy of radiation is related to the temperature of the source of radiant energy. But the emitted radiation is not confined to a single wavelength but is spread over a *continuous spectrum*. *How is this total energy distributed amongst the different wavelengths?* The attempts to find an answer to this question constitute a long and painstaking history of the growth and development of physics, culminating into the revolutionary concept of *quanta* by Planck.

Considered from quantum theory, the problem now appears to be rather easy. It essentially consists in finding out *how many quanta* having energy lying between ϵ *and* $\epsilon + d\epsilon$ (or frequency between ν and $\nu + d\nu$) are contained *per unit volume* in an enclosure containing black radiation at *temperature* T, analogous to finding the distribution of energy amongst the molecules. But this is the present day knowledge !

Before the advent of the quantum theory, the workers in the field like Wien, Rayleigh, Jeans, Planck were *not aware of the mechanism of emission and absorption of radiation*. In fact, this mechanism came to be known only through Planck's investigation with black radiation. They had at their disposal only the *laws of classical thermodynamics* and *electromagnetism*. It would be both interesting and instructive to follow how they arrived at a complete solution of the problem, step by step, with much toil and labour.

The *first step* in this direction was taken by Wien in 1893. He showed that the spectral distribution of energy emitted by a black body at a temperature T can be expressed as

$$u_\lambda d\lambda = C\lambda^{-5} f(\lambda T) d\lambda \qquad (12.11.1)$$

where $u_\lambda d\lambda$ is the energy density of radiation between the wavelengths λ and $\lambda + d\lambda, C$ is a constant and $f(\lambda T)$ is a function of the product λT.

The above relation is known as **Wien's law of energy distribution.**

Wien also showed that if *radiation of a particular wavelength at a certain temperature is adiabatically changed to another wavelength, then the temperature changes in the inverse ratio*. that is,

$$\lambda_1 T_1 = \lambda_2 T_2 = \lambda_3 T_3 = \cdots$$

$$\text{or,} \quad \lambda T = \text{constant} \qquad (12.11.2)$$

This relation is known as **Wien's displacement law.**

12.11.1 Derivation of Wien's law

We shall first derive *Wien's displacement law* purely from thermodynamic consideration, and then the *Wien's law of spectral distribution of energy*.

Wien's displacement law–Consider an imaginary experiment performed with a spherical enclosure with perfectly but *diffusely* (angle of incidence $\neq$ angle of reflection) reflecting wall, capable of moving slowly outward. Let the enclosure be filled with *black radiation* characterised by thermodynamic temperature T and be allowed to *expand adiabatically* by supposing its walls moving out slowly with a small uniform velocity $v (\ll c)$. Due to the reversible adiabatic expansion, the temperature T will fall to say T', but it can be proved thermodynamically that the *quality of radiation will remain unaffected*, the radiation will remain as *black radiation*, characteristic of the lower temperature (**Note 2**).

Although the nature of the radiation will not change due to expansion, the wavelength will get altered due to *Doppler shift* on each reflection at the moving wall. Now, for normal incidence, a particular wave of wavelength λ changes, due to Doppler principle, to

$$\lambda' = \lambda\frac{c+v}{c-v} \simeq \lambda\left(1+\frac{2v}{c}\right) \quad [\because\ v \ll c] \quad \text{for each reflection.}$$

When the incidence is at an angle α with the normal, the resolved part of v along the direction of the ray is effective in changing the wavelength.

$\therefore$ For *oblique* incidence, $\lambda' = \lambda\left(1+\dfrac{2v\cos\alpha}{c}\right)$.

For a narrow beam incident on the boundary of the spherical enclosure at B making an angle α with the radius at B, the angle of reflection however will be different from α, since the walls reflect *diffusely*.

If the ray starts from A (Fig.12.14) at $t = 0$ and $s_1, s_2, \cdots, s_n$ are the lengths of the chords travelled by radiation between successive reflections in unit time,

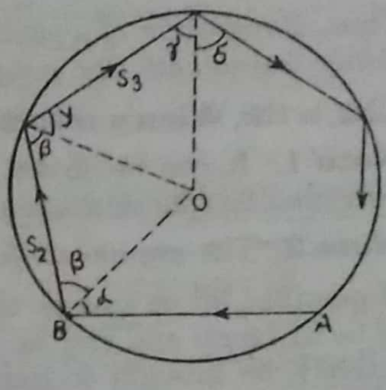

Fig.12.14. Change in wavelength of radiation on diffusely reflecting wall

$$s_1 + s_2 + \cdots + s_n = \sum_{i=1}^{n} s_i = c$$

and, $\cos\alpha = s_1/2r$, $\cos\beta = s_2/2r$, etc.

$$\therefore \quad \text{After the first reflection, } \lambda_1 = \lambda\left(1+\frac{2s_1}{2r}\frac{v}{c}\right) = \lambda\left(1+\frac{s_1 v}{rc}\right)$$

After n successive reflections, the final λ-value is

$$\lambda_n = \lambda_f = \lambda\left(1+\frac{s_1 v}{rc}\right)\left(1+\frac{s_2 v}{rc}\right)\cdots\left(1+\frac{s_n v}{rc}\right)$$

$$\simeq \lambda\left(1+\frac{v}{rc}\sum_{i=1}^{n} s_i\right) = \lambda\left(1+\frac{vc}{rc}\right) = \lambda\left(1+\frac{v}{r}\right) \qquad (12.11.3)$$

$$\therefore \quad \lambda_f - \lambda = \frac{d\lambda}{dt} = \lambda\frac{v}{r} = \frac{\lambda}{r}\frac{dr}{dt} \quad \left(\because\ v = \frac{dr}{dt}\right)$$

$$\therefore \quad \frac{d\lambda}{\lambda} = \frac{dr}{r} \qquad (12.11.4)$$

[Note. On integrating (12.11.4) we obtain $\lambda/r =$ const., or $\lambda^3/V =$ constant.]

Since the given amount of energy, we started with, is now distributed over a larger volume and a part is spent in doing external work — the *work done by radiation pressure* on the walls — the temperature of the enclosure will change. The process being adiabatic,

$$dQ = dU + pdV = 0$$

where dU is the total change in internal energy and pdV the work done by radiant energy. where u is the energy density of radiation.

But $U = uV$ and $p = \frac{1}{3}u$ where u is the energy density of radiation.

$$\therefore \quad d(uV) + \tfrac{1}{3}udV = 0$$

$$\text{or,} \quad \frac{du}{u} + \frac{4}{3}\frac{dV}{V} = 0, \text{ on simplification.}$$

Integrating, we obtain $$uV^{4/3} = \text{constant} \tag{12.11.5}$$

But from Stefan's law, $u = \alpha T^4$ and $V = \frac{4}{3}\pi r^3$.

Eq. (12.11.5) thus yields : $$rT = \text{const.} \quad \therefore \quad \frac{dT}{T} = -\frac{dr}{r} \tag{12.11.6}$$

Combining (12.11.5) and (12.11.4), $\frac{d\lambda}{\lambda} = -\frac{dT}{T} \Rightarrow \lambda T = \text{const.}$, on integration,

Thus, *if radiation of a particular wavelength at a given temperature is adiabatically changed to another wavelength, the temperature changes in the inverse ratio.*

This is the **Wien's displacement law** of radiation.

Note 1. It can be shown that the results will hold good for an enclosure of any shape, not restricted to spherical one only.

Note 2. The *arguments for the quality of radiation remaining unaffected* are :

If possible, let us assume that the radiation, on expansion, changes character. Specifically, let *red* be in *excess* and *blue* be in *deficit.* Let two bodies R and B, absorbing red and blue rays respectively, be brought to temperature of the enclosure and introduced into it. Then R will be more heated and B cooled. By working a Carnot engine between them, the excess heat in R may be converted to work and bring the quality to full or black radiation. The temperature will be lowered below T' since some energy is converted to work. The bodies R and B are now removed and the enclosure compressed adiabatically till the original volume is restored. The temperature at the end of the process will be less than T' and, by previous supposition, the radiation again will *not* be full radiation. By inserting absorbing bodies and working a Carnot cycle, the temperature can be lowered till the quality is again full or black radiation. Hence, finally the original volume is colder and some work is obtained during the whole process. Thus we may get work continuously by using the heat of a single body (reservoir) without having to maintain a second body at a lower temperature. This is against the Second law. Hence our assumption is *not true.*

Note 3. Eq. (12.11.5) $uV^{4/3}$ = const. may also be written as $pV^{4/3}$ = const., for an *adiabatic process.* Thus radiation behaves *like a perfect gas* with $\gamma = 4/3$. But *do not get tempted* to conclude that the ratio of heat capacities for a photon gas is 4/3.

Wien's distribution law — We shall *first* prove that $u_\lambda \lambda^5$ = const. By isolating waves of lengths between λ and $\lambda + d\lambda$ in the spherical chamber and subjecting those waves only to an *adiabatic expansion*, the work done is $\frac{1}{3}u_\lambda d\lambda.\Delta V$. This must be equal to the decrease in the total energy content, that is, $-\Delta(u_\lambda d\lambda.V)$.

$$\begin{aligned}
\therefore \quad \tfrac{1}{3}u_\lambda d\lambda.\Delta V &= -\Delta(u_\lambda d\lambda \cdot V) \\
&= -(\Delta u_\lambda)d\lambda \cdot V - u_\lambda \Delta(d\lambda.V) \\
&= -\Delta u_\lambda d\lambda.V - u_\lambda \Delta(d\lambda)V - u_\lambda d\lambda(\Delta V)
\end{aligned}$$

Dividing throughout by $u_\lambda V.d\lambda$, we have

$$\frac{1}{3}\frac{\Delta V}{V} = -\frac{\Delta u_\lambda}{u_\lambda} - \frac{\Delta(d\lambda)}{d\lambda} - \frac{\Delta V}{V}$$

$$\text{or,} \quad \frac{4}{3}\frac{\Delta V}{V} = -\frac{\Delta u_\lambda}{u_\lambda} - \frac{\Delta\lambda}{\lambda} \tag{12.11.7}$$

since $d\lambda$ changes in the same way as λ, (as proved in the box below) $\Delta(d\lambda)/d\lambda = \Delta\lambda/\lambda$.

Proof — For an adiabatic process in an elcosure, λT = constant (adiabatic invariant). Let $d\lambda$ = spectral range about λ, and $\Delta\lambda$ = change in λ due to adiabatic process.

$$\therefore \quad \lambda T = (\lambda + \Delta\lambda)T' \text{ when } \lambda \to (\lambda + \Delta\lambda)$$

and $$(\lambda + d\lambda)T = (\lambda + d\lambda + \Delta\lambda + \Delta d\lambda)T', \text{ when } (\lambda + d\lambda) \to (\lambda + d\lambda + \Delta\lambda + \Delta d\lambda)$$

$$\therefore \quad \frac{T}{T'} = \frac{\lambda + \Delta\lambda}{\lambda} = \frac{\lambda + d\lambda + \Delta\lambda + \Delta d\lambda}{\lambda + d\lambda} = \frac{d\lambda + \Delta d\lambda}{d\lambda}$$

$$\therefore \quad \frac{\lambda + \Delta\lambda}{\lambda} = \frac{d\lambda + \Delta d\lambda}{d\lambda} \text{ or, } \frac{\Delta\lambda}{\lambda} = \frac{\Delta(d\lambda)}{d\lambda}$$

which simply proves that λ changes in the same way as $d\lambda$ does in an adiabatic process.

Again, since $V = \frac{4}{3}\pi r^3 \Rightarrow \Delta V = 4\pi r^2 \Delta r$.

$$\therefore \quad \frac{\Delta r}{r} = \frac{1}{3}\frac{\Delta V}{V} = \frac{\Delta\lambda}{\lambda}, \quad \text{using (12.11.4).} \qquad (12.11.8)$$

From (12.11.7), using (12.11.8), we get

$$5\frac{\Delta\lambda}{\lambda} + \frac{\Delta u_\lambda}{u_\lambda} = 0$$

$$\therefore \quad u_\lambda \lambda^5 = \text{const.} = u_{\lambda'} \lambda'^5 \qquad (12.11.9)$$

where $u_{\lambda'}$, is the density of radiation λ' to which λ is transformed by expansion. This will correspond to equilibrium radiation-density at temperature $T' = \lambda T/\lambda'$.

Plainly, u_λ is a function of T. So the constant in (12.11.9) must involve T and must remain constant throughout the adiabatic change of wavelength. But, λT = constant, for this process. So this constant is a function of the product λT.

$$\therefore \quad u_\lambda d\lambda = \frac{C}{\lambda^5} f(\lambda T) d\lambda \qquad (12.11.10)$$

This is the common expression for **Wien's distribution law.**

Eq. (12.11.10) can be given another form as shown

$$u_\lambda d\lambda = \frac{C}{\lambda^5} f(\lambda T) d\lambda = \frac{CT^5}{(\lambda T)^5} f(\lambda T) d\lambda$$

$$= CT^5 F(\lambda T) d\lambda \qquad (12.11.11)$$

where the function $F(\lambda T) = (\lambda T)^{-5} f(\lambda T)$.

From (12.11.11), we obtain

$$\frac{u_\lambda}{u_{\lambda'}} = \left(\frac{T}{T'}\right)^5$$

The *radiation density* thus *increases directly as the fifth power of absolute temperature* an important information.

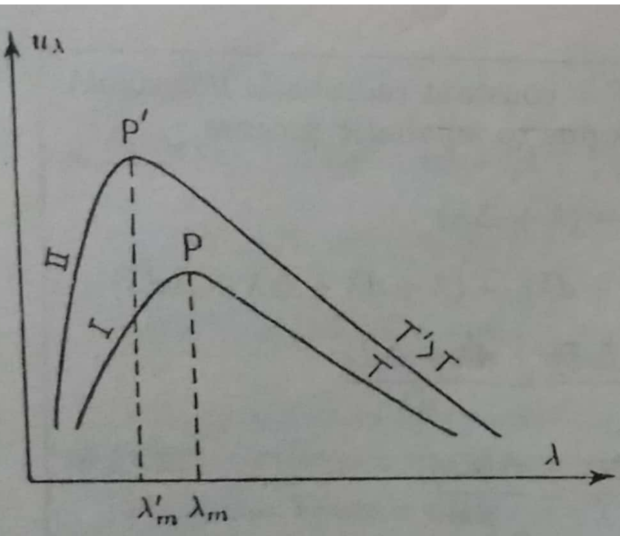

Fig.12.15. Correspondence of black body radiation curves at different temperatures

If the distribution at a certain temperature T be given (Fig.12.15 : Curve I) that at another higher temperature T' can be obtained graphically by simply shortening the abscissa in the ratio (T/T') and enlarging the corresponding ordinate in the ratio $(T'/T)^5$. Consequently, the curve becomes higher, but more closed up, and the total area representing the intensity is changed in the ratio $(T'/T)^4$. The wavelength-interval $(d\lambda)$ is also shortened in the ratio (T/T'). The point P in the curve I refers to maximum emission. Corresponding to P, there is a point P' in curve II such that

$$\lambda_m T = \lambda'_m T' = \text{const.} = b$$

where λ_m's correspond to the wavélength of maximum emission, and b is the root of the equation $du_\lambda/d\lambda = 0$.

12.12.1 Rayleigh-Jeans' law

In the light of the failure of Wien's radiation law to account for the experimental results at longer wavelengths, Rayleigh and Jeans approached the problem differently and came up with what is known as *Rayleigh-Jeans law.*

The Rayleigh-Jeans law of spectral distribution of black radiation is derived on the basis of the following two theorems :

(i) the theorem of stationary waves in a hollow enclosure, and

(ii) the theorem of equipartition of energy.

The black body chamber is filled with *diffuse* radiation of all frequencies between 0 and ∞. This radiation is composed of e.m waves in space. They are reflected time and again from the walls forming stationary waves in space of the enclosure. Rayleigh showed that the number of possible independent vibrations between the frequencies ν and $\nu + d\nu$ per unit volume is proportional to $\nu^2 d\nu$. This has just been proved in Art. 12.12.

Once this is established, the distribution of energy can be dervied, by the principle of equipartition. Considering the kinetic and the potential energy, the energy of each vibration will be $\frac{1}{2}kT + \frac{1}{2}kT = kT$.

$$\therefore \quad u_\nu d\nu = \frac{8\pi\nu^2 d\nu}{c^3} kT \tag{12.12.6}$$

In terms of wavelength, (12.12.6) becomes

$$u_\lambda d\lambda = \frac{8\pi kT}{\lambda^4} d\lambda \tag{12.12.7}$$

This is the **Rayleigh-Jeans** law of spectral distribution of energy.

12.13 Planck's law of radiation

The laws of radiation as deduced theoretically from classical principle by Wien and Rayleigh-Jeans however *failed to interpret the experimentally observed energy distribution amongst the different wavelengths of black radiation.* In this context, Planck questioned the validity of the very assumptions of the classical theory on which the deductions were based.

From Wien's law, u vanishes for $\lambda = 0$ or $\lambda = \infty$ as it should. But it also makes u finite for $T = \infty$ which contradicts T^4-law of Stefan. In a like manner, Rayleigh-Jeans law indicates that the energy radiated in a given wavelength range increases as λ decreases and tends to ∞ for short wavelengths. But this is *not true* experimentally. Further, if $u_\lambda d\lambda$ is integrated from 0 to ∞ assuming *continuous emission of energy*, the total energy except for $T = 0$ turns out to be *infinite*! This is a sheer absurdity and is known as **ultraviolet catastrophe**.

In this background, Planck put forward his daring hypothesis of *discontinuous process of exchange of energy* and arrived at the most satisfactory formula, both on theoretical and experimental grounds, to solve the problem of distribution of energy. Planck's law of radiation can be written as

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

The equation agrees with the energy curves obtained experimentally by various workers and presents the Wien's law and the Rayleigh-Jeans law as its special cases.

Deduction of Planck's law — The most elegant and straight-forward method for deducing Planck's radiation law is to consider that a chamber containing black radiation is full of *photons in molecular chaos* and determine *how the energy is distributed amongst the photons.* This method of deduction is due to S. N. Bose and is described in the Chapter on '*Statistical Mechanics*'.

We shall however consider here the original deduction by Planck himself when the quantum theory of radiation was unknown. In fact, it was Planck's investigation on black radiation that gave birth to it. Further, there were other problems as well in those days : (i) the notion of temperature in radiation problems could be introduced only by considering energy exchanges between matter and radiation, and (ii) the Bohr's concept on the mechanism of emission and absorption of radiation was not known; the radiation and gas molecules could not exchange energy directly. Planck therefore (a) imagined that a black radiation chamber is filled up not only by radiation but by ideal gas molecules as well and (b) introduced resonators of molecular dimensions as *via media* between radiation and gas moelcules; the resonators absorb energy from radiation, and transfer a part or whole of it to molecules during collision and the thermodynamic equlibrium is thereby established.

The resonators introduced by Planck were *electric dipoles* having motion along the fixed axis, the centre of mass of each remaining *stationary*. The motion is supposed to be *simple harmonic* with a natural frequency ν.

Planck *gave up* the *classical hypothesis of continuous emission* of radiation by resonators and assumed that they emit energy only when the energy absorbed has a certain *minimum value* ϵ *or some integral multiple* of it. Thus radiation of energy ϵ can be obtained from resonators having the energy content $\epsilon, 2\epsilon, 3\epsilon, \cdots r\epsilon, \cdots$ etc.

Let us now compute the mean energy of these resonators. Using Maxwell's classical formula, the probability that the resonator will possess the energy E is $e^{-E/kT}$. Therefore, the mean energy of a resonator $\bar{\epsilon}$ is given by

$$\bar{\epsilon} = \frac{\sum_0^\infty n\epsilon e^{-n\epsilon/kT}}{\sum_0^\infty e^{-n\epsilon/kT}} = \frac{\sum_0^\infty n\epsilon e^{-\beta n\epsilon}}{\sum_0^\infty e^{-\beta n\epsilon}}, \quad \text{where} \quad \beta = 1/kT$$

$$= -\frac{d}{d\beta} \ln \sum_0^\infty e^{-\beta n\epsilon} = -\frac{d}{d\beta} \ln \frac{1}{1 - e^{-\beta\epsilon}}$$

$$= \frac{\epsilon e^{-\beta\epsilon}}{1 - e^{-\beta\epsilon}} = \frac{\epsilon}{e^{\beta\epsilon} - 1} = \frac{\epsilon}{e^{\epsilon/kT} - 1}$$

and *not* kT as given by the equipartition law.

Planck puts $\epsilon \propto \nu$ or $\epsilon = h\nu$, where h is Planck's constant.

$$\therefore \quad \bar{\epsilon} = \frac{h\nu}{e^{h\nu/kT} - 1} \tag{12.13.1}$$

But the number of modes of vibration per unit volume between the frequency range ν and $\nu + d\nu$ is, as already shown,

$$\frac{8\pi\nu^2 d\nu}{c^3}$$

So, the energy density of radiation in the frequency interval ν and $\nu + d\nu$ is

$$u_\nu d\nu = \frac{8\pi\nu^2 d\nu}{c^3} \times \frac{h\nu}{e^{h\nu/kT} - 1}$$

$$= \frac{8\pi h}{c^3} \frac{\nu^3}{(e^{h\nu/kT} - 1)} d\nu \tag{12.13.2}$$

which is the well-known **Planck's law** of radiation.

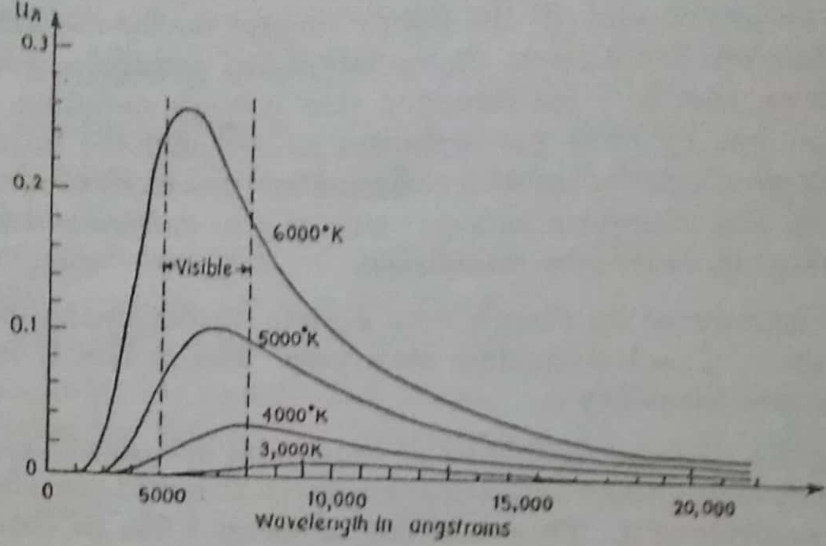

Fig.12.18. Energy distribution in the spectrum of a black body

Expressed in wavelength, using, $\nu = c/\lambda \Rightarrow d\nu = \left|\frac{c}{\lambda^2}d\lambda\right|$, we obtain

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{ch/\lambda kT}-1}d\lambda \tag{12.13.3}$$

which is an *alternative form* of Planck's law that gives energy density in the wavelength range λ and $\lambda + d\lambda$ in the spectrum of a black body.

12.13.1 Deduction from Planck's law

The different classical laws of radiation such as Wien's law, Rayleigh-Jean's law etc. follow readily from Planck's law and is discussed below.

Wien's law — In terms of wavelength λ, the Planck's formula runs as

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT}-1} \tag{12.13.4}$$

For short wavelength (or high frequency) and low temperature, λT is small so that the exponential term in the denominator has a value much greater than 1. Thus from (12.13.4)

$$u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT}} = C_1\lambda^{-5}e^{-C_2/\lambda T}d\lambda \tag{12.13.5}$$

where the constants $C_1 = 8\pi hc$ and $C_2 = hc/k$.

This (12.13.5) is *Wien's radiation law*, an essentially empirical formula containing two adjustable constants C_1 and C_2. Wien chose these constants so that the fit obtained by him was rather good *except* at long wavelengths.

Rayleigh-Jeans law — For long wavelength (or low frequency) and high temperature $hc/\lambda kT \ll 1$.

$$\therefore \text{ From (12.13.4)}, \quad u_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{\left(1+\frac{hc}{\lambda kT}+\cdots\right)-1} = \frac{8\pi kT}{\lambda^4}d\lambda \tag{12.13.6}$$

which is *Rayleigh-Jeans law*. It shows that the *energy density of radiation is inversely proportional to the fourth power of* λ.

Wien's displacement law — Differentiating the right hand member of equation (12.13.4), with respect to λ and setting the result to zero, we obtain the λ (at temperature T) for which the energy density is maximum. If λ_m symbolizes this particular λ, then

$$-\frac{5 \times 8\pi hc}{\lambda_m^6 (e^{hc/\lambda_m kT} - 1)} + \frac{8\pi hc}{\lambda_m^5} \cdot \frac{e^{hc/\lambda_m kT}}{(e^{hc/\lambda_m kT} - 1)^2} \cdot \frac{hc}{kT\lambda_m^2} = 0.$$

Introducing $x = hc/\lambda_m kT$ and eliminating the common factors, we get

$$\frac{xe^x}{e^x - 1} = 5 \qquad (12.13.7)$$

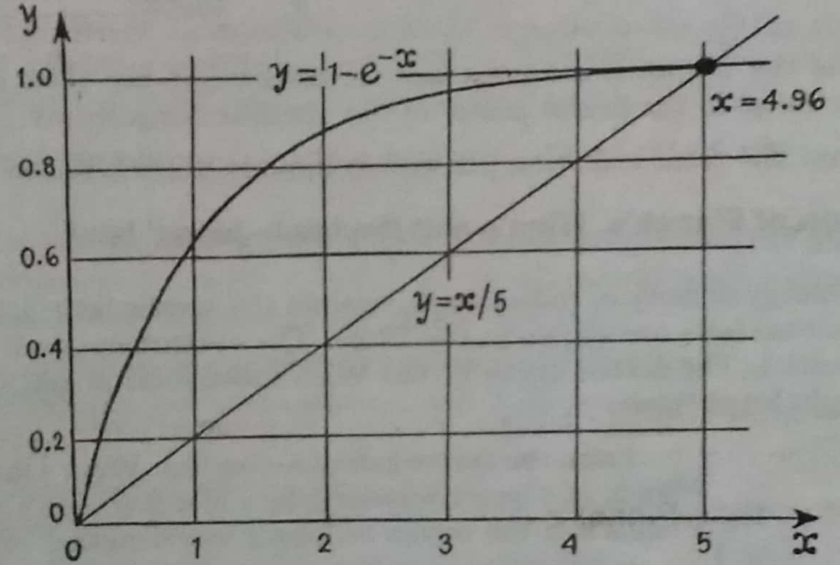

Fig.12.19. Graphical solution of the equation $xe^x/(e^x - 1) = 5$

This transcendental equation that can be solved graphically or numerically shows at once that there must be a root in the neighbourhood of 5. Applying the usual method, the exact value is $x = 4.965$.

$$\therefore \quad \frac{hc}{\lambda_m kT} = 4.965$$

$$\text{or,} \quad \lambda_m T = \frac{hc}{4.965k} = 0.2896 \text{ cm.K}$$

$$= \text{constant} \qquad (12.13.8)$$

This is *Wien's displacement law.* It states that *as the temperature of a black body is increased, the position of maximum emission moves in the direction of shorter waves in such a way that the product* $\lambda_m T$ = *constant.* This provides us with a *simple method of determining the temperature* of the outer surface of all radiating bodies, including the heavenly ones.

Stefan-Boltzmann law — The T^4-law of Stefan-Boltzmann also follows from Planck's. The total radiation Q is obtained by integrating $u_\nu d\nu$ for waves of all frequencies from 0 to ∞. Thus the total radiation emitted by a black body is

$$Q = \int_0^\infty u_\nu d\nu = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu$$

From Wien's law for maximum u_λ, $\partial u_\lambda/\partial\lambda = 0$. If that λ be symbolised by λ_m,

$$-\frac{5}{\lambda_m} + \frac{x}{\lambda_m} = 0 \qquad \therefore \quad x = 5$$

$$\text{or,} \quad \frac{ch}{\lambda_m kT} = 5 \qquad \therefore \quad \lambda_m T = \frac{ch}{5k} = 0.287\text{cm K}$$

From Rayleigh's law similarly, by imposing $\partial u_\lambda/\partial\lambda = 0$ at $\lambda = \lambda_m$,

$$-3(8\pi kT)\lambda_m^{-5} = 0$$

$\therefore$ Either $T = 0$ or $\lambda = \infty$. But *neither* of these is *possible.* So, of the three formulae, this suffers from the *most serious drawback.* But the *agreement between* that of Planck and Wien, in this respect, *is excellent* as both yield almost identical values for $\lambda_m T$.

12.16 Illustrated examples

Ex. 1. *Each sq. metre of the sun's surface radiates energy at the rate of* $6.3 \times 10^7 J/m^2/s$ *and Stefan constant is* $5.669 \times 10^{-8} W/m^2/K^4$. *Find the temperature of the sun's surface, assuming that the Stefan's law applies to the radiation.*

Solution. From Stefan's law, we have $Q = \sigma T^4$.

Here, by the problem, $Q = 6.3 \times 10^7 J/m^2/s$ and $\sigma = 5.669 \times 10^{-8} W/m^2/K^4$.

$$\therefore\ T^4 = \frac{Q}{\sigma} = \frac{6.3 \times 10^7}{5.669 \times 10^{-8}} K^4 \quad \text{or,} \quad T = 5773K$$

$\therefore$ Temperature of the sun's surface = **5773K**

Ex. 2. *Calculate the pressure exerted by sunlight on the earth's surface, given that the solar constant is* $8.148 J/cm^2/min$.

Solution. Here, I = intensity of radiation = $8.148 \times 10^4/60 J/m^2/s$

The solar rays incident on the earth's surface are parallel. So the pressure of radiation

$$p = \frac{I}{c} = \frac{8.148 \times 10^4}{60 \times 3 \times 10^8} = \mathbf{0.0452 \times 10^{-4} N/m^2}$$

Ex. 3. *Assuming the earth to be a spherical black body moving in a circular orbit of radius* 1.5×10^8 km *round the sun, find its equilibrium temperature if the sun is considered to be a sphere of radius* 7×10^5 km *radiating as a black body at a temperature of* 6×10^3 K. *Assume that the radius of the earth is small compared to the radius of its orbit.* **(Calcutta Hons.)**

Solution. Here, by the problem

$$R = \text{radius of earth's orbit} = 1.5 \times 10^8 \text{km}$$
$$R_s = \text{radius of the sun} = 7 \times 10^5 \text{km}$$
$$R_e = \text{radius of the earth } (R_e \ll R)$$

Now, the energy radiated per sec from the sun,

$$Q = \sigma \times 4\pi R_s^2 \times (6 \times 10^3)^4 \qquad \text{(i)}$$

The above energy is contained within a sphere of radius R. So, the energy received by the earth per second is

$$\frac{Q}{4\pi R^2} \times \pi R_e^2 = \frac{QR_e^2}{4R^2} = 4\pi\sigma R_s^2 \times (6)^4 \times 10^{12}\frac{R_e^2}{4R^2}, \text{ using (i).}$$

For thermal equilibrium, the rate at which the energy is received must be equal to the rate at which the energy is emitted.

$$\therefore \quad 4\pi R_e^2 \times \sigma T_e^4 = 4\pi\sigma R_s^2 \times (6)^4 \times 10^{12}\frac{R_e^2}{4R^2}$$

$$\therefore \quad T_e^4 = \left(\frac{R_s}{2R}\right)^2 \times (6)^4 \times 10^{12}$$

$$\therefore \quad T_e = 6 \times 10^3 \times \left(\frac{7 \times 10^5}{2 \times 1.5 \times 10^8}\right)^{\frac{1}{2}} = \mathbf{290K}$$

Ex. 4. *The temperature of a body falls from* 30°C *to* 20°C *in* 5 minutes. *The air temperature is* 13°C. *Find the temperature of the body after another* 5 minutes.

Solution. We have, from Newton's law, the rate of cooling, $-d\theta/dt = C(\theta - \theta_s)$

$$\text{or,} \quad \frac{d\theta}{\theta - \theta_s} = -Cdt$$

Integrating,

$$\ln(\theta - \theta_s) = -Ct + A (= a \text{ const.}) \qquad \text{(ii)}$$

At $t = 0$, $\theta = 30°$C and θ_s = surrounding temperature = 13°C.

$$\therefore \quad \ln(30 - 13) = A = \ln 17$$

When $t = 5$ min, $\theta = 20°$C, $\theta_s = 13°$C. Therefore, from (ii)

$$\ln(20 - 13) = -C \times 5 + \ln 17 \quad \therefore \quad -C = \frac{1}{5}\ln\frac{7}{17}$$

After another 5 min., $t = 10$ min. (from start), $\theta_s = 13°$C, $\theta = ?$

$$\therefore \quad \ln(\theta - 13) = -C \times 10 + \ln 17 = 2\ln\frac{7}{17} + \ln 17$$

$$\therefore \quad \theta - 13 = \left(\frac{7}{17}\right)^2 \times 17 = \frac{49}{17} \quad \therefore \quad \theta = 13 + \frac{49}{17} = \mathbf{15.88°C}$$

Ex. 5. *Calculate the black body temperature of sun from the following data : Stefan constant* $= 1.37 \times 10^{-12}\,\text{cal/cm}^2/\text{s}$, *solar constant* $= 2.3\,\text{cal/cm}^2/\text{min}$, *radius of the sun* $= 7 \times 10^{10}$ cm, *distance between the sun and the earth* $= 1.5 \times 10^{13}$ cm. **(Delhi Hons.)**

Solution. Here, r = radius of the sun $= 7 \times 10^{10}$ cm. R = distance between the sun and the earth $= 1.5 \times 10^{13}$ cm, σ = Stefan constant $= 1.37 \times 10^{12}\,\text{cal/cm}^2/\text{s}$ and S = solar constant $= 2.3\,\text{cal/cm}^2/\text{min}$.

$$\therefore \quad Q = \frac{4\pi R^2 S}{4\pi r^2} = \left(\frac{R}{r}\right)^2 S$$

But, by Stefan's law, $Q = \sigma T_s^4$, where T_s = temperature of the sun. $\therefore \; \sigma T_s^4 = (R/r)^2 S$,

$$\therefore \quad T_s^4 = \left(\frac{R}{r}\right)^2 \frac{S}{\sigma} = \left(\frac{1.5 \times 10^{13}}{7 \times 10^{10}}\right)^2 \times \frac{2.3}{60 \times 1.37 \times 10^{-12}} = (5987)^4 \text{K}^4.$$

$\therefore$ Temperature of the sun, $T_s = \mathbf{5987\,K}$

Ex. 6. *An aluminium foil of relative emittance* 0.1 *is placed between two concentric spheres (assumed perfectly black radiators) at temperatures* 300 K *and* 200 K *respectively. Find the temperature of the foil after the steady state is reached and the rate of energy transfer between one of the spheres and the foil* ($\sigma = 5.672 \times 10^{-8}$ SI *unit*).

Solution. Here $T_1 = 300$ K, $T_2 = 200$ K, $e = 0.1$ and $\sigma = 5.672 \times 10^{-8}$ SI.

Let, in the steady state, x be the temperature in degree Kelvin of the foil.

$$\therefore \quad e\sigma(T_1^4 - x^4) = e\sigma(x^4 - T_2^4)$$

$$\text{or,} \quad (300)^4 - x^4 = x^4 - (200)^4$$

$$\therefore \quad x^4 = 48.5 \times 10^8; \quad \therefore \quad x = \mathbf{263.9\,K}$$

$$Q = e\sigma(T_1^4 - x^4)$$

$$= 0.1 \times 5.672 \times 10^{-8}\left[(300)^4 - (263.9)^4\right] = \mathbf{18.4\,W/m^2}$$

Ex. 7. *A black body at a temperature of* 1646 K *has the wavelength corresponding to the maximum emission* (λ_m) *equal to* 1.78 micron. *Find the temperature of the moon (assumed to be a black body) if* λ_m *for the moon is* 14 micron.

Solution. From Wien's displacement law, $\lambda_m T$ = const.

Here $(\lambda_m)_1 = 1.78$ micron, $T_1 = 1646$ K; $(\lambda_m)_2 = 14$ micron and $T_2 = ?$

$$\therefore \quad 1.78 \times 1646 = 14 \times T_2$$

$$\therefore \quad T_2 = \frac{1.78 \times 1646}{14} = \mathbf{209.28\,K}$$

$\therefore$ Temperature of the moon $= \mathbf{209.28\,K}$

Ex. 8. *When a total radiation pyrometer in a room with surroundings at* 300 K *is sighted on a black body at* 600 K, *the deflection observed in the galvanometer is* 6 *divisions. When sighted on another black body, the deflection is found to be* 400 *divisions. Find the temperature of the latter.*

Solution. If d be the deflection of the galvanometer, T_o the temperature of the surroundings and T the temperature of the black body,

$$d = K(T^4 - T_o^4), \quad K = \text{const.}$$

Now, $d_1 = 6, d_2 = 400, T_o = 300$ K. Let T_2 be the required temperature.

$$\therefore \quad 6 = K(600^4 - 300^4); \quad 400 = K(T_2^4 - 300^4) \qquad \text{(i)}$$

Dividing one by the other in (i) and re-arranging

$$400 = \frac{6}{600^4 - 300^4}(T_2^4 - 300^4)$$

$$\therefore \quad T_2^4 = \frac{400}{6}(600^4 - 300^4) + 300^4 = 1687^4 + 300^4; \quad \therefore \quad T_2 = \mathbf{1687.38\ K}$$

Ex. 9. *Assume the sun to be a black body at temperature* 5800 K. *Use Stefan's law to show that the total radiant energy emitted by sun per second is* 3.95×10^{26} J. *Also show that the rate at which energy is reaching the top of the earth's atmosphere is* $1.4\ \text{kW.m}^{-2}$.

(Delhi Hons.)

Solution. From the given data, we have $T = 5800$ K

Now, the radius of the sun, $r = 7 \times 10^8$ m

$\therefore$ Surface area of the sun, $A = 4\pi r^2 = 4 \times 3.142 \times (7 \times 10^8)^2 \text{m}^2$

Stefan constant, $\sigma = 5.672 \times 10^{-8}$ SI unit.

$\therefore$ Using Stefan's law, the total energy Q emitted by the sun is given by

$$U = A\sigma T^4 = 4\pi r^2 \sigma T^4$$

$$= 4 \times 3.142 \times (7 \times 10^8)^2 \times 5.672 \times 10^{-8} \times (5800)^4$$

$$= \mathbf{3.95 \times 10^{26} J}$$

The distance of the earth's atmosphere from the sun, $r_1 = 1.5 \times 10^{11}$m

$\therefore$ Solar energy reaching unit area of earth's atmosphere per sec is

$$Q_e = \frac{Q}{4\pi r_1^2} = \frac{3.95 \times 10^{26}}{4 \times 3.142 \times (1.5 \times 10^{11})^2}$$

$$= 1.4 \times 10^3 \text{W/m}^2$$

$$= \mathbf{1.4\ kWm^{-2}}$$