Semester-IV

B.Sc (Honours) in Physics

C8T: Mathematical Physics III

Lecture

on

Eigen-values and Eigenvector

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Lecture-V

(Specially: Diagonalization of Matrices)

Syllabus

- ❑ Eigen-values and Eigenvectors
- ❑Cayley- Hamiliton Theorem.
- ❑ Diagonalization of Matrices.
- ❑ Solutions of Coupled Linear Ordinary Differential Equations.
- ❑ Functions of a Matrix.

Diagonalization of Matrices

Here we explain how to diagonalize a matrix. We only describe the procedure of diagonalization, and no justification will be given. The process can be summarized as follows. A concrete example is provided below, and several exercise problems has been discussed in this lecture.

Diagonalization Procedure

Example of a matrix diagonalization

Step 1: Find the characteristic polynomial

Step 2: Find the eigenvalues

Step 3: Find the eigenspaces

Step 4: Determine linearly independent eigenvectors

Step 5: Define the invertible matrix S

Step 6: Define the diagonal matrix D

Step 7: Finish the diagonalization

Diagonalization Problems and Examples

A Hermitian Matrix can be diagonalized by a unitary matrix

More diagonalization problems

The general procedure of the diagonalization is explained in the lecture "How to Diagonalize a Matrix. Step by Step Explanation"

Diagonalization Procedure

Let **A** be the **n×n** matrix that you want to diagonalize (if possible).

1. Find the characteristic polynomial **p(t)** of **A**.
2. Find eigen values **λ** of the matrix **A** and their algebraic multiplicities from the characteristic polynomial **p(t)**.
3. For each eigen value **λ** of **A**, find a basis of the eigen space $\mathbf{E}_\lambda$.
4. If there is an eigen value **λ** such that the geometric multiplicity of **λ**, dim($\mathbf{E}_\lambda$), is less than the algebraic multiplicity of **λ**, then the matrix **A** is not diagonalizable. If not, **A** is diagonalizable, and proceed to the next step.
5. If we combine all basis vectors for all eigenspaces, we obtained n linearly independent eigenvectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n$.
6. Define the nonsingular matrix $\mathbf{S} = [\mathbf{v}_1\mathbf{v}_2\ldots\mathbf{v}_n]$.
7. Define the diagonal matrix **D**, whose (**i**,**i**)-entry is the eigen value **λ** such that the **i**-th column vector $\mathbf{v}_i$ is in the eigen space $\mathbf{E}_\lambda$.
8. Then the matrix **A** is diagonalized as $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$

Example of a matrix diagonalization

Now let us examine these steps with an example.
Let us consider the following 3×3 matrix.

$$A = \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$$

We want to diagonalized matrix if possible.

Step 1: Find the characteristic polynomial

The characteristic polynomial p(t) of A is

$$P(t) = \det(A - tI) = \begin{bmatrix} 4-t & -3 & -3 \\ 3 & -2-t & -3 \\ -1 & 1 & 2-t \end{bmatrix}$$

Using the cofactor expansion, we get
$p(t) = -(t-1)^2(t-2)$.

Step 2: Find the eigen values

From the characteristic polynomial obtained in Step 1, we see that eigen values are
$\lambda=1$ with algebraic multiplicity 2 and
$\lambda=2$ with algebraic multiplicity 1.

Step 3: Find the eigen spaces

Let us first find the eigen space E_1 corresponding to the eigen value $\lambda=1$.
By definition, E_1 is the null space of the matrix

$$\text{A-I} = \begin{bmatrix} 3 & -3 & -3 \\ 3 & -3 & -3 \\ -1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

by elementary row operations.
Hence if (A−I)x=0(A−I)x=0 for x∈R3x∈R3, we have
x1=x2+x3

$$E_1 = N(A - 1) = \left\{ x \in R^3 \mid x = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

From this, we see that the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigen space E_1.

Thus, the dimension of E_1, which is the geometric multiplicity of $\lambda=1$, is 2. Similarly, we find a basis of the eigen space $E_2=N(A-2I)$ for the eigen value $\lambda=2$. We have

$$A - 2I = \begin{bmatrix} 2 & -3 & -3 \\ 3 & -4 & -3 \\ -1 & -4 & -3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

by elementary row operations.

Then if $(A-2I)x=0$ for $x \in R^3$, then we have

$x_1=-3x_3$ and $x_2=-3x_3$.

$$E_2 = N(A - 21) = \left\{ x \in R^3 \mid x = x_3 \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix} \right\}$$

From this we see that the set $\left\{\begin{bmatrix}-3\\-3\\1\end{bmatrix}\right\}$

is a basis for the eigen space E_2 and the geometric multiplicity is 1.

Since for both eigen values, the geometric multiplicity is equal to the algebraic multiplicity, the matrix **A** is not defective, and hence diagonalizable.

Step 4: Determine linearly independent eigenvectors

From Step 3, the vectors $V_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, V_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \; V_3 = \begin{bmatrix}-3\\-3\\1\end{bmatrix}$

are linearly independent eigenvectors.

Step 5: Define the invertible matrix S

Define the matrix $S=[v_1 v_2 v_3]$. Thus we have

$S = \begin{bmatrix}1 & 1 & -3\\1 & 0 & -3\\0 & 1 & 1\end{bmatrix}$ and the matrix S is nonsingular (since the column vectors are linearly independent).

Step 6: Define the diagonal matrix D

Define the diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Note that (1,1)-entry of D is 1 because the first column vector

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} of\ S\ is\ in\ the\ eigen\ space\ E_1, that\ is, v_1\ is\ an$

$eigen\ vector\ corresponding\ to\ eigen\ value\ \lambda = 1$

Similarly, the (2,2)-entry of D is 1 because the second column

$$V_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} of\ S\ is\ in\ E_1.$$

The (3,3)-entry of D is 2 because the third column vector

$$V_3 = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix} of\ S\ is\ in\ E_2.$$

(The order you arrange the vectors v1,v2,v3 to form S does not matter but once you made S, then the order of the diagonal entries is determined by S, that is, the order of eigenvectors in S.)

Step 7: Finish the diagonalization

Finally, we can diagonalize the matrix **A** as $\mathbf{S^{-1}AS = D}$, where

$$S = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

If S is orthogonal then $S^{-1} = S^T$

Some problems will be discuss in next class…..