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8.2.3 Conservation of Momentum

According to Newton's second law, the force on an object is equal to the rate of change of its momentum:

$$\mathbf{F} = \frac{d\mathbf{p}_{\text{mech}}}{dt}.$$

Equation 8.22 can therefore be written in the form

$$\frac{d\mathbf{p}_{\text{mech}}}{dt} = -\epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \mathbf{S}\,d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a}, \tag{8.28}$$

where $\mathbf{p}_{\text{mech}}$ is the total (mechanical) momentum of the particles contained in the volume $\mathcal{V}$. This expression is similar in structure to Poynting's theorem (Eq. 8.9), and it invites an analogous interpretation: The first integral represents *momentum stored in the electromagnetic fields themselves*:

$$\mathbf{p}_{\text{em}} = \mu_0\epsilon_0 \int_{\mathcal{V}} \mathbf{S}\,d\tau, \tag{8.29}$$

while the second integral is the *momentum per unit time flowing in through the surface*. Equation 8.28 is the general statement of *conservation of momentum* in electrodynamics: Any increase in the *total* momentum (mechanical plus electromagnetic) is equal to the momentum brought in by the fields. (If $\mathcal{V}$ is *all* of space, then *no* momentum flows in or out, and $\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{em}}$ is constant.)

As in the case of conservation of charge and conservation of energy, conservation of momentum can be given a differential formulation. Let $\boldsymbol{\wp}_{\text{mech}}$ be the density of *mechanical* momentum, and $\boldsymbol{\wp}_{\text{em}}$ the density of momentum in the fields:

$$\boxed{\boldsymbol{\wp}_{\text{em}} = \mu_0\epsilon_0\mathbf{S}.} \tag{8.30}$$

Then Eq. 8.28, in differential form, says

$$\boxed{\frac{\partial}{\partial t}(\boldsymbol{\wp}_{\text{mech}} + \boldsymbol{\wp}_{\text{em}}) = \nabla \cdot \overleftrightarrow{\mathbf{T}}.} \tag{8.31}$$

Evidently $-\overleftrightarrow{\mathbf{T}}$ is the **momentum flux density**, playing the role of $\mathbf{J}$ (current density) in the continuity equation, or $\mathbf{S}$ (energy flux density) in Poynting's theorem. Specifically, $-T_{ij}$ is the momentum in the i direction crossing a surface oriented in the j direction, per unit area, per unit time. Notice that the Poynting vector has appeared in two quite different roles: $\mathbf{S}$ itself is the energy per unit area, per unit time, transported by the electromagnetic fields, while $\mu_0\epsilon_0\mathbf{S}$ is the momentum per unit volume stored in those fields. Similarly, $\overleftrightarrow{\mathbf{T}}$ plays a dual role: $\overleftrightarrow{\mathbf{T}}$ itself is the electromagnetic stress (force per unit area) acting on a surface, and $-\overleftrightarrow{\mathbf{T}}$ describes the flow of momentum (the momentum current density) transported by the fields.

8.2.4 Angular Momentum

By now the electromagnetic fields (which started out as mediators of forces between charges) have taken on a life of their own. They carry *energy* (Eq. 8.13)

$$u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right), \tag{8.32}$$

and *momentum* (Eq. 8.30)

$$\boldsymbol{\wp}_{em} = \mu_0\epsilon_0\mathbf{S} = \epsilon_0(\mathbf{E}\times\mathbf{B}), \tag{8.33}$$

and, for that matter, *angular* momentum:

$$\boldsymbol{\ell}_{em} = \mathbf{r}\times\boldsymbol{\wp}_{em} = \epsilon_0\left[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})\right]. \tag{8.34}$$

Even perfectly static fields can harbor momentum and angular momentum, as long as $\mathbf{E}\times\mathbf{B}$ is nonzero, and it is only when these field contributions are included that the classical conservation laws hold.

9.2.3 Energy and Momentum in Electromagnetic Waves

According to Eq. 8.13, the energy per unit volume stored in electromagnetic fields is

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right). \tag{9.53}$$

In the case of a monochromatic plane wave (Eq. 9.48)

$$B^2 = \frac{1}{c^2}E^2 = \mu_0\epsilon_0 E^2, \tag{9.54}$$

so the *electric and magnetic contributions are equal*:

$$u = \epsilon_0 E^2 = \epsilon_0 E_0^2\cos^2(kz - \omega t + \delta). \tag{9.55}$$

As the wave travels, it carries this energy along with it. The energy flux density (energy per unit area, per unit time) transported by the fields is given by the Poynting vector (Eq. 8.10):

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}). \tag{9.56}$$

For monochromatic plane waves propagating in the z direction,

$$\mathbf{S} = c\epsilon_0 E_0^2\cos^2(kz - \omega t + \delta)\,\hat{\mathbf{z}} = cu\,\hat{\mathbf{z}}. \tag{9.57}$$

Notice that $\mathbf{S}$ is the energy density (u) times the velocity of the waves ($c\,\hat{\mathbf{z}}$)—as it *should* be. For in a time Δt, a length $c\,\Delta t$ passes through area A (Fig. 9.12), carrying with it an energy $uAc\,\Delta t$. The energy per unit time, per unit area, transported by the wave is therefore uc.

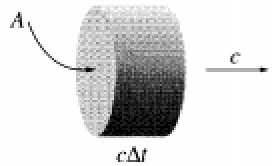

Figure 9.12

Electromagnetic fields not only carry *energy*, they also carry *momentum*. In fact, we found in Eq. 8.30 that the momentum density stored in the fields is

$$\boldsymbol{\wp} = \frac{1}{c^2}\mathbf{S}. \tag{9.58}$$

For monochromatic plane waves, then,

$$\boldsymbol{\wp} = \frac{1}{c}\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{\mathbf{z}} = \frac{1}{c}u\,\hat{\mathbf{z}}. \tag{9.59}$$

In the case of *light*, the wavelength is so short ($\sim 5 \times 10^{-7}$ m), and the period so brief ($\sim 10^{-15}$ s), that any macroscopic measurement will encompass many cycles. Typically, therefore, we're not interested in the fluctuating cosine-squared term in the energy and momentum densities; all we want is the *average* value. Now, the average of cosine-squared over a complete cycle[6] is $\frac{1}{2}$, so

$$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2, \tag{9.60}$$

$$\langle \mathbf{S} \rangle = \frac{1}{2}c\epsilon_0 E_0^2\,\hat{\mathbf{z}}, \tag{9.61}$$

$$\langle \boldsymbol{\wp} \rangle = \frac{1}{2c}\epsilon_0 E_0^2\,\hat{\mathbf{z}}. \tag{9.62}$$

I use brackets, $\langle\ \rangle$, to denote the (time) average over a complete cycle (or *many* cycles, if you prefer). The average power per unit area transported by an electromagnetic wave is called the **intensity**:

$$I \equiv \langle S \rangle = \frac{1}{2}c\epsilon_0 E_0^2. \tag{9.63}$$

[6]There is a cute trick for doing this in your head: $\sin^2\theta + \cos^2\theta = 1$, and over a complete cycle the average of $\sin^2\theta$ is equal to the average of $\cos^2\theta$, so $\langle \sin^2 \rangle = \langle \cos^2 \rangle = 1/2$. More formally,

$$\frac{1}{T}\int_0^T \cos^2(kz - 2\pi t/T + \delta)\,dt = 1/2.$$

When light falls on a perfect absorber it delivers its momentum to the surface. In a time Δt the momentum transfer is (Fig. 9.12) $\Delta \mathbf{p} = \langle \wp \rangle A c\, \Delta t$, so the **radiation pressure** (average force per unit area) is

$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}. \tag{9.64}$$

(On a perfect *reflector* the pressure is *twice* as great, because the momentum switches direction, instead of simply being absorbed.) We can account for this pressure qualitatively, as follows: The electric field (Eq. 9.48) drives charges in the x direction, and the magnetic field then exerts on them a force ($q\bar{\mathbf{v}}\mathbf{x}\,\bar{\mathbf{B}}$) in the z direction. The net force on all the charges in the surface produces the pressure.

9.2 Electromagnetic Waves in Vacuum

9.2.1 The Wave Equation for E and B

In regions of space where there is no charge or current, Maxwell's equations read

$$\left.\begin{array}{llll} \text{(i)} & \nabla\cdot\mathbf{E} = 0, & \text{(iii)} & \nabla\times\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla\cdot\mathbf{B} = 0, & \text{(iv)} & \nabla\times\mathbf{B} = \mu_0\epsilon_0\dfrac{\partial \mathbf{E}}{\partial t}. \end{array}\right\} \tag{9.40}$$

They constitute a set of coupled, first-order, partial differential equations for **E** and **B**. They can be *de*coupled by applying the curl to (iii) and (iv):

$$\begin{aligned} \nabla\times(\nabla\times\mathbf{E}) &= \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = \nabla\times\left(-\frac{\partial \mathbf{B}}{\partial t}\right) \\ &= -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) = -\mu_0\epsilon_0\frac{\partial^2 \mathbf{E}}{\partial t^2}, \\ \nabla\times(\nabla\times\mathbf{B}) &= \nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B} = \nabla\times\left(\mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \\ &= \mu_0\epsilon_0\frac{\partial}{\partial t}(\nabla\times\mathbf{E}) = -\mu_0\epsilon_0\frac{\partial^2 \mathbf{B}}{\partial t^2}. \end{aligned}$$

Or, since $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$,

$$\boxed{\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.} \tag{9.41}$$

We now have *separate* equations for $\mathbf{E}$ and $\mathbf{B}$, but they are of *second* order; that's the price you pay for decoupling them.

In vacuum, then, each Cartesian component of $\mathbf{E}$ and $\mathbf{B}$ satisfies the **three-dimensional wave equation,**

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}.$$

(This is the same as Eq. 9.2, except that $\partial^2 f / \partial z^2$ is replaced by its natural generalization, $\nabla^2 f$.) So Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, traveling at a speed

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \text{ m/s}, \tag{9.42}$$

which happens to be precisely the velocity of light, c. The implication is astounding: Perhaps light *is* an electromagnetic wave.[3] Of course, this conclusion does not surprise anyone today, but imagine what a revelation it was in Maxwell's time! Remember how ϵ_0 and μ_0 came into the theory in the first place: they were constants in Coulomb's law and the Biot-Savart law, respectively. You measure them in experiments involving charged pith balls, batteries, and wires—experiments having nothing whatever to do with light. And yet, according to Maxwell's theory you can calculate c from these two numbers. Notice the crucial role played by Maxwell's contribution to Ampère's law ($\mu_0 \epsilon_0 \partial \mathbf{E} / \partial t$); without it, the wave equation would not emerge, and there would be no electromagnetic theory of light.

).3 Electromagnetic Waves in Matter

9.3.1 Propagation in Linear Media

Inside matter, but in regions where there is no *free* charge or *free* current, Maxwell's equations become

$$\left.\begin{array}{llll} \text{(i)} & \nabla \cdot \mathbf{D} = 0, & \text{(iii)} & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla \cdot \mathbf{B} = 0, & \text{(iv)} & \nabla \times \mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t}. \end{array}\right\} \tag{9.65}$$

If the medium is *linear*,

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B}, \tag{9.66}$$

and *homogeneous* (so ϵ and μ do not vary from point to point), Maxwell's equations reduce to

$$\left.\begin{array}{lll}\text{(i)} & \nabla\cdot\mathbf{E}=0, & \text{(iii)} \quad \nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla\cdot\mathbf{B}=0, & \text{(iv)} \quad \nabla\times\mathbf{B}=\mu\epsilon\dfrac{\partial\mathbf{E}}{\partial t},\end{array}\right\} \tag{9.67}$$

which (remarkably) differ from the vacuum analogs (Eqs. 9.40) only in the replacement of $\mu_0\epsilon_0$ by $\mu\epsilon$.[7] Evidently electromagnetic waves propagate through a linear homogeneous medium at a speed

$$v=\frac{1}{\sqrt{\epsilon\mu}}=\frac{c}{n}, \tag{9.68}$$

where

$$n\equiv\sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \tag{9.69}$$

is the **index of refraction** of the material. For most materials, μ is very close to μ_0, so

$$n\cong\sqrt{\epsilon_r}, \tag{9.70}$$

where ϵ_r is the dielectric constant (Eq. 4.34). Since ϵ_r is almost always greater than 1, light travels *more slowly* through matter—a fact that is well known from optics.

All of our previous results carry over, with the simple transcription $\epsilon_0\to\epsilon$, $\mu_0\to\mu$, and hence $c\to v$ (see Prob. 8.15). The energy density is[8]

$$u=\frac{1}{2}\left(\epsilon E^2+\frac{1}{\mu}B^2\right), \tag{9.71}$$

and the Poynting vector is

$$\mathbf{S}=\frac{1}{\mu}(\mathbf{E}\times\mathbf{B}). \tag{9.72}$$

For monochromatic plane waves the frequency and wave number are related by $\omega=kv$ (Eq. 9.11), the amplitude of $\mathbf{B}$ is $1/v$ times the amplitude of $\mathbf{E}$ (Eq. 9.47), and the intensity is

$$I=\frac{1}{2}\epsilon vE_0^2. \tag{9.73}$$

The interesting question is this: What happens when a wave passes from one transparent medium into another—air to water, say, or glass to plastic? As in the case of waves on a string, we expect to get a reflected wave and a transmitted wave. The details depend on the exact nature of the electrodynamic boundary conditions, which we derived in Chapter 7 (Eq. 7.64):

$$\left.\begin{array}{ll}\text{(i)} \quad \epsilon_1E_1^{\perp}=\epsilon_2E_2^{\perp}, & \text{(iii)} \quad \mathbf{E}_1^{\parallel}=\mathbf{E}_2^{\parallel}, \\ \text{(ii)} \quad B_1^{\perp}=B_2^{\perp}, & \text{(iv)} \quad \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel}=\dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel}.\end{array}\right\} \tag{9.74}$$

These equations relate the electric and magnetic fields just to the left and just to the right of the interface between two linear media. In the following sections we use them to deduce the laws governing reflection and refraction of electromagnetic waves.

Problem Base questions:

Problem 9.10 The intensity of sunlight hitting the earth is about 1300 $\mathrm{W/m^2}$. If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?

Problem 9.11 In the complex notation there is a clever device for finding the time average of a product. Suppose $f(\mathbf{r}, t) = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_a)$ and $g(\mathbf{r}, t) = B\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_b)$. Show that $\langle fg\rangle = (1/2)\mathrm{Re}(\tilde{f}\tilde{g}^*)$, where the star denotes complex conjugation. [Note that this only works if the two waves have the same $\mathbf{k}$ and ω, but they need not have the same amplitude or phase.] For example

$$\langle u\rangle = \frac{1}{4}\mathrm{Re}\left(\epsilon_0\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right) \quad \text{and} \quad \langle\mathbf{S}\rangle = \frac{1}{2\mu_0}\mathrm{Re}(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*).$$

Problem 9.12 Find all elements of the Maxwell stress tensor for a monochromatic plane wave traveling in the z direction and linearly polarized in the x direction (Eq. 9.48). Does your answer make sense? (Remember that $\overleftrightarrow{\mathbf{T}}$ represents the momentum flux density.) How is the momentum flux density related to the energy density, in this case?

References:

- *Griffiths D.J. Introduction to electrodynamics_2*