- SEMESTER –VI (HONOURS)
- PAPER : CC13T(EM THEORY)
- TOPIC: EM WAVES IN CONDUCTORS

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Electromagnetic wave in conductor:

Inside a conductor, free charges can move/migrate around in response to *EM* fields contained therein, as we saw for the case of the longitudinal $\vec{E}$-field inside a current-carrying wire that had a static potential difference ΔV across its ends. Even in the static case of electric charge residing on the surface of a conductor, we saw that $\vec{E}_{inside}(\vec{r}) = 0$, but recall that this actually means (as we showed last semester) that the ***NET*** electric field inside the conductor is zero, *i.e.* $\vec{E}^{NET}_{inside}(\vec{r}) = 0$.

n.b. here, we assume {for simplicity's sake} that the conductor is linear/homogeneous/isotropic – *i.e.* no crystalline structure/no anisotropies/no inhomogenities/voids/defects…

From Ohm's Law, we know that the ***free*** current density $\vec{J}_{free}(\vec{r},t)$ is proportional to the (ambient) electric field inside the conductor: $\boxed{\vec{J}_{free}(\vec{r},t) = \sigma_C \vec{E}(\vec{r},t)}$ where:

σ_C = ***conductivity*** of the metal conductor $(Siemens/m = Ohm^{-1}/m)$ and $\boxed{\sigma_C = 1/\rho_C}$

ρ_C = ***resistivity*** of the metal conductor $(Ohm\text{-}m)$.

Thus inside such a conductor, we can assume that the linear/homogeneous/isotropic conducting medium has electric permittivity ε and magnetic permeability μ. Maxwell's equations inside such a conductor {with $\vec{J}_{free}(\vec{r},t) \neq 0$} are thus:

Using Ohm's Law: $\vec{J}_{free}(\vec{r},t) = \sigma_C \vec{E}(\vec{r},t)$

1) $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t) = \rho_{free}(\vec{r},t)/\varepsilon}$ 2) $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t) = 0}$

3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{\partial \vec{B}(\vec{r},t)}{\partial t}}$ 4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu\vec{J}_{free}(\vec{r},t) + \mu\varepsilon\frac{\partial \vec{E}(\vec{r},t)}{\partial t} = \mu\sigma_C\vec{E}(\vec{r},t) + \mu\varepsilon\frac{\partial \vec{E}(\vec{r},t)}{\partial t}}$

Electric charge is (always) conserved, thus the continuity equation inside the conductor is:

$\boxed{\vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t) = -\frac{\partial \rho_{free}(\vec{r},t)}{\partial t}}$ but: $\boxed{\vec{J}_{free}(\vec{r},t) = \sigma_C\vec{E}(\vec{r},t)}$

$\therefore \boxed{\sigma_C\left(\vec{\nabla}\cdot\vec{E}(\vec{r},t)\right) = -\frac{\partial \rho_{free}(\vec{r},t)}{\partial t}}$ but: $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t) = \frac{1}{\varepsilon}\rho_{free}(\vec{r},t)}$

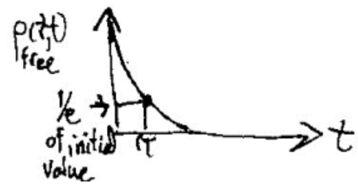
thus: $\boxed{\frac{\sigma_C\rho_{free}(\vec{r},t)}{\varepsilon} = -\frac{\partial \rho_{free}(\vec{r},t)}{\partial t}}$ or: $\boxed{\frac{\partial \rho_{free}(\vec{r},t)}{\partial t} + \left(\frac{\sigma_C}{\varepsilon}\right)\rho_{free}(\vec{r},t) = 0}$ ⇐ 1st order linear, homogeneous differential equation

The general solution of this differential equation for the free charge density is of the form:

$$\boxed{\rho_{free}(\vec{r},t) = \rho_{free}(\vec{r},t=0)e^{-\sigma_C t/\varepsilon} = \rho_{free}(\vec{r},t=0)e^{-t/\tau_{relax}}}$$ *i.e.* a ***damped*** exponential !!!

Characteristic ***damping*** time: $\boxed{\tau_{relax} \equiv \varepsilon/\sigma_C}$ = ***charge relaxation time*** {*aka* ***time constant***}.

Thus, the continuity equation $\vec{\nabla}\bullet\vec{J}_{free}(\vec{r},t) = -\partial\rho_{free}(\vec{r},t)/\partial t$ inside a conductor tells us that any free charge density $\rho_{free}(\vec{r},t=0)$ initially present at time $t = 0$ is ***exponentially*** damped / dissipated in a characteristic time $\boxed{\tau_{relax} \equiv \varepsilon/\sigma_C}$ = ***charge relaxation time*** {*aka* ***time constant***}, such that at when: $t = \tau_{relax} \equiv \varepsilon/\sigma_C$: $\boxed{\rho_{free}(\vec{r},t=\tau_{relax}) = \rho_{free}(\vec{r},t=0)e^{-1} = 0.369\cdot\rho_{free}(\vec{r},t=0)}$



Calculation of the Charge Relaxation Time for Pure Copper:

$$\boxed{\rho_{Cu} = 1/\sigma_{Cu} = 1.68\times10^{-8}\,\Omega\text{-m}} \;\Rightarrow\; \boxed{\sigma_{Cu} = 1/\rho_{Cu} = 5.95\times10^{7}\ \text{Siemens/m}}$$

If we assume $\varepsilon_{Cu} \approx 3\varepsilon_o = 3\times8.85\times10^{-8}$ F/m for copper metal, then:

$$\boxed{\tau_{Cu}^{relax} = \varepsilon_{Cu}/\sigma_{Cu} = \rho_{Cu}\,\varepsilon_{Cu} = 4.5\times10^{-19}\ \text{sec}} \ \ !!!$$

However, the characteristic (*aka* mean} ***collision time*** of free electrons in pure copper is $\tau_{Cu}^{coll} \simeq \lambda_{Cu}^{coll}/v_{thermal}^{Cu}$ where $\lambda_{Cu}^{coll} \simeq 3.9\times10^{-8}\,m$ = mean free path (between successive collisions) in pure copper, and $v_{thermal}^{Cu} \simeq \sqrt{3k_BT/m_e} \simeq 12\times10^{5}\ m/sec$ and thus we obtain: $\tau_{coll}^{Cu} \simeq 3.2\times10^{-13}$ sec.

Hence we see that the calculated charge relaxation time in pure copper, $\tau_{Cu}^{relax} \simeq 4.5\times10^{-19}$ sec is $\ll$ than the calculated collision time in pure copper, $\tau_{coll}^{Cu} \simeq 3.2\times10^{-13}$ sec.

Furthermore, the ***experimentally measured*** charge relaxation time in pure copper is $\tau_{Cu}^{relax}(\text{exp't}) \simeq 4.0\times10^{-14}$ sec, which is ≈ 5 orders of magnitude ***larger*** than the ***calculated*** charge relaxation time $\tau_{Cu}^{relax} \simeq 4.5\times10^{-19}$ sec. The problem here is that {the ***macroscopic***} Ohm's Law is simply out of its range of validity on such short time scales! Two ***additional*** facts here are that ***both*** ε and σ_C are ***frequency-dependent*** quantities { *i.e.* $\varepsilon = \varepsilon(\omega)$ and $\sigma_C = \sigma_C(\omega)$ }, which becomes ***increasingly*** important at the higher frequencies $(f = 2\pi/\omega \sim 1/\tau_{relax})$ associated with short time-scale, transient-type phenomena!

So in reality, if we are willing to wait a short time (*e.g.* $\Delta t \sim 1\,\text{ps} = 10^{-12}$ sec) then, any initial free charge density $\rho_{free}(\vec{r},t=0)$ accumulated inside a ***good*** conductor at $t = 0$ will have dissipated away/damped out, and from that time onwards, $\rho_{free}(\vec{r},t) = 0$ ***can*** be safely assumed.

Note: For a ***poor*** conductor $(\sigma_C \to 0)$, then: $\tau_{relax} \equiv \varepsilon/\sigma_C \to \infty$!!! Please keep this in mind…

After ***many*** charge relaxation time constants, *e.g.* $20\tau_{relax} \le \Delta t \simeq 1\ ps = 10^{-12}\ sec$, Maxwell's ***steady-state*** equations for a ***good*** conductor become {with $\rho_{free}(\vec{r}, t \ge \Delta t) = 0$ from then onwards}:

Maxwell's equations for a ***charge-equilibrated*** conductor

1) $\vec{\nabla} \bullet \vec{E}(\vec{r},t) = 0$ 2) $\vec{\nabla} \bullet \vec{B}(\vec{r},t) = 0$

3) $\vec{\nabla} \times \vec{E}(\vec{r},t) = -\dfrac{\partial \vec{B}(\vec{r},t)}{\partial t}$ 4) $\vec{\nabla} \times \vec{B}(\vec{r},t) = \mu\sigma_C \vec{E}(\vec{r},t) + \mu\varepsilon \dfrac{\partial \vec{E}(\vec{r},t)}{\partial t} = \mu\left(\sigma_C \vec{E}(\vec{r},t) + \varepsilon \dfrac{\partial \vec{E}(\vec{r},t)}{\partial t}\right)$

These equations are different from the previous derivation(s) of monochromatic plane *EM* waves propagating in free space/vacuum and/or in linear/homogeneous/isotropic non-conducting materials {*n.b.* only equation 4) has changed}, hence we re-derive {***steady-state***} wave equations for $\vec{E}$ & $\vec{B}$ from scratch. As before, we apply $\vec{\nabla} \times (\ )$ to equations 3) and 4):

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$
$$= \vec{\nabla}(\cancel{\vec{\nabla} \bullet \vec{E}}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu\sigma_C \vec{E} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t}\right)$$
$$= \nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma_C \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu\left(\sigma_C (\vec{\nabla} \times \vec{E})\right) + \varepsilon \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$
$$= \vec{\nabla}(\cancel{\vec{\nabla} \bullet \vec{B}}) - \nabla^2 \vec{B} = -\mu\sigma_C \frac{\partial \vec{B}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$
$$= \nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma_C \frac{\partial \vec{B}}{\partial t}$$

<u>Again</u>: $\nabla^2 \vec{E}(\vec{r},t) = \mu\varepsilon \dfrac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} + \mu\sigma_C \dfrac{\partial \vec{E}(\vec{r},t)}{\partial t}$ <u>and</u>: $\nabla^2 \vec{B}(\vec{r},t) = \mu\varepsilon \dfrac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2} + \mu\sigma_C \dfrac{\partial \vec{B}(\vec{r},t)}{\partial t}$

Note that the {***steady-state***} 3-D wave equations for $\vec{E}$ and $\vec{B}$ in a conductor have an additional term that has a single time derivative – which is analogous *e.g.* to a velocity-dependent ***damping term*** associated with the motion of a 1-D mechanical harmonic oscillator.

The general solution(s) to the above {***steady-state***} wave equations are usually in the form of an oscillatory function × a damping term (*i.e.* a decaying exponential) – in the direction of the propagation of the *EM* wave, complex plane-wave type solutions for $\vec{E}$ and $\vec{B}$ associated with the above wave equation(s) are of the general form:

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_o e^{i(\tilde{k}z - \omega t)} \quad \text{and:} \quad \tilde{\vec{B}}(\vec{r},t) = \tilde{\vec{B}}_o e^{i(\tilde{k}z - \omega t)} = \frac{1}{\tilde{v}} \hat{k} \times \tilde{\vec{E}}(\vec{r},t) = \left(\frac{\tilde{k}}{\omega}\right) \hat{k} \times \tilde{\vec{E}}(\vec{r},t)$$

n.b. with {frequency-dependent} ***complex*** wave number: $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$

where: $k(\omega) = \Re e\{\tilde{k}(\omega)\}$ and $\kappa(\omega) = \Im m\{\tilde{k}(\omega)\}$ and corresponding ***complex*** wave vector $\tilde{\vec{k}}(\omega) = \tilde{k}(\omega)\hat{k} = \tilde{k}(\omega)\hat{z}$ (for *EM* wave propagating in the $\hat{k} = +\hat{z}$ direction, ***here***).

Physically, $k(\omega) = \Re e\{\tilde{k}(\omega)\}$ is associated with wave ***propagation***, and $\kappa(\omega) = \Im m\{\tilde{k}(\omega)\}$ is associated with wave ***attenuation*** (*i.e.* ***dissipation***).

We plug $\tilde{\vec{E}}(\vec{r},t)=\tilde{\vec{E}}_o e^{i(\tilde{k}z-\omega t)}$ and $\tilde{\vec{B}}(\vec{r},t)=\tilde{\vec{B}}_o e^{i(\tilde{k}z-\omega t)}$ into their respective wave equations above, and obtain from each wave equation the same/identical ***characteristic*** ***equation*** – {*aka* a ***dispersion*** ***relation***} between complex $\tilde{k}(\omega)$ and ω {please work this out yourselves!}:

$$\boxed{\tilde{k}^2(\omega)=\mu\varepsilon\omega^2+i\mu\sigma_C\omega}$$

Thus, since $\tilde{k}(\omega)=k(\omega)+i\kappa(\omega)$, then:

$$\boxed{\tilde{k}^2(\omega)=\left(k(\omega)+i\kappa(\omega)\right)^2=k^2(\omega)-\kappa^2(\omega)+2ik(\omega)\kappa(\omega)=\mu\varepsilon\omega^2+i\mu\sigma_C\omega}$$

If we {temporarily} suppress the ω-dependence of complex $\tilde{k}(\omega)$, this relation becomes:

$$\boxed{\tilde{k}^2=(k+i\kappa)^2=k^2-\kappa^2+2ik\kappa=\mu\varepsilon\omega^2+i\mu\sigma_C\omega}$$

We can re-write this expression as: $\boxed{\left[\left(k^2-\kappa^2\right)-\mu\varepsilon\omega^2\right]+i\left[2k\kappa-\mu\sigma_C\omega\right]=0}$, which ***must*** be true for ***any/all*** values of {any of} the parameters involved. The only in-general way that this relation can hold is if ***both*** $\left[\left(k^2-\kappa^2\right)-\mu\varepsilon\omega^2\right]=0$ **.*and*.** $\left[2k\kappa-\mu\sigma_C\omega\right]=0$. Then:

$$\boxed{k^2-\kappa^2=\mu\varepsilon\omega^2} \quad \text{and:} \quad \boxed{2k\kappa=\mu\sigma_C\omega}$$

Thus, we have ***two*** separate/independent equations: $k^2-\kappa^2=\mu\varepsilon\omega^2$ and: $2k\kappa=\mu\sigma_C\omega$. We have ***two*** unknowns: k and κ. Hence, we solve these equations ***simultaneously*** to determine k and κ!

From the ***latter*** relation, we see that: $\boxed{\kappa=\frac{1}{2}\mu\sigma_C\omega/k}$. Plug ***this*** result into the ***other*** relation:

$$\boxed{k^2-\kappa^2=k^2-\left(\tfrac{1}{2}\mu\sigma_C\omega/k\right)^2=k^2-\frac{1}{k^2}\left(\tfrac{1}{2}\mu\sigma_C\omega\right)^2=\mu\varepsilon\omega^2}$$

Then multiply by k^2 and rearrange the terms to obtain the following relation:

$$\boxed{k^4-\left(\mu\varepsilon\omega^2\right)k^2-\left(\tfrac{1}{2}\mu\sigma_C\omega\right)^2=0}$$

This may ***look*** like a scary equation to try to solve (*i.e.* a ***quartic*** equation - *eeekkk*!), but it's actually just a ***quadratic*** equation! {So, it's really just a ***leprechaun***, masquerading as a ***unicorn***!}

Define: $x\equiv k^2$, $a\equiv 1$, $b\equiv-\left(\mu\varepsilon\omega^2\right)$ and $c\equiv-\left(\frac{1}{2}\mu\sigma_C\omega\right)^2$, this equation then becomes "the usual" quadratic equation, of the form: $ax^2+bx+c=0$, with solution(s)/root(s):

$$\boxed{x=\frac{-b\mp\sqrt{b^2-4ac}}{2a}} \quad \text{or:} \quad \boxed{k^2=\frac{1}{2}\left[+\left(\mu\varepsilon\omega^2\right)\mp\sqrt{\left(\mu\varepsilon\omega^2\right)^2+4\left(\tfrac{1}{2}\mu\sigma_C\omega\right)^2}\right]}$$

This relation can be re-written as:

$$k^2 = \frac{1}{2}(\mu\varepsilon\omega^2)\left[1 \mp \sqrt{1 + 4\frac{(\mu^2\sigma_C^2\omega^2)}{4(\mu^2\varepsilon^2\omega^{4\,2})}}\right] = \frac{1}{2}(\mu\varepsilon\omega^2)\left[1 \mp \sqrt{1 + \frac{(\sigma_C^2)}{(\varepsilon^2\omega^2)}}\right] = \frac{1}{2}(\mu\varepsilon\omega^2)\left[1 \mp \sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]$$

On ***physical*** grounds ($k^2 > 0$), we ***must*** select the + sign, hence:

$$k^2 = \frac{1}{2}(\mu\varepsilon\omega^2)\left[1 + \sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]$$ and thus: $$k = \sqrt{k^2} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[1 + \sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right]^{1/2} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2} + 1\right]^{1/2}$$

Having thus solved for k (or equivalently, k^2), we can use ***either*** of our original ***two*** relations to solve for κ, *e.g.* $k^2 - \kappa^2 = \mu\varepsilon\omega^2$, thus:

$$\kappa^2 = k^2 - \mu\varepsilon\omega^2 = \frac{1}{2}(\mu\varepsilon\omega^2)\left[1 + \sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2}\right] - \mu\varepsilon\omega^2 = \frac{1}{2}(\mu\varepsilon\omega^2)\left[\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2} - 1\right]$$

Hence {finally}, we obtain:

$$k(\omega) = \Re e\{\tilde{k}(\omega)\} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2} + 1\right]^{1/2}$$ and: $$\kappa(\omega) = \Im m\{\tilde{k}(\omega)\} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}$$

The above two relations ***clearly*** show the frequency dependence of ***both*** the ***real*** and ***imaginary*** components of the complex wavenumber $\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$. This physically means that *EM* wave propagation in a conductor is ***dispersive*** (*i.e. EM* wave propagation is ***frequency dependent***).

Note also that the ***imaginary*** part of $\tilde{k}(\omega)$, $\kappa(\omega) = \Im m\{\tilde{k}(\omega)\}$ results in an ***exponential attenuation***/***damping*** of the monochromatic plane *EM* wave with increasing *z*:

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_o e^{i(\tilde{k}z - \omega t)} = \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz - \omega t)}$$ where: $$\tilde{k}(\omega) = k(\omega) + i\kappa(\omega)$$

and: $$\tilde{\vec{B}}(\vec{r},t) = \tilde{\vec{B}}_o e^{i(\tilde{k}z - \omega t)} = \tilde{\vec{B}}_o e^{-\kappa z} e^{i(kz - \omega t)} = \frac{\tilde{k}}{\omega}\hat{k} \times \tilde{\vec{E}}(z,t) = \frac{\tilde{k}}{\omega}\hat{k} \times \tilde{\vec{E}}_o e^{-\kappa z} e^{i(kz - \omega t)}$$

The ***characteristic distance*** *z* over which $\vec{E}$ and $\vec{B}$ are attenuated/reduced to $1/e = e^{-1} = 0.368$ of their initial values (at $z = 0$) is known as the ***skin depth***, $\delta_{sc}(\omega) \equiv 1/\kappa(\omega)$ (<u>SI units</u>: meters).

i.e. $$\delta_{sc}(\omega) = \frac{1}{\kappa(\omega)} = \frac{1}{\omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma_C}{\varepsilon\omega}\right)^2} - 1\right]^{1/2}} \Rightarrow \begin{aligned} \tilde{\vec{E}}(z = \delta_{sc}, t) &= \tilde{\vec{E}}_o e^{-1} e^{i(kz - \omega t)} \\ \tilde{\vec{B}}(z = \delta_{sc}, t) &= \tilde{\vec{B}}_o e^{-1} e^{i(kz - \omega t)} \end{aligned}$$

The ***real*** part of $\tilde{k}(\omega)$, *i.e.* $k(\omega) = \Re e\{\tilde{k}(\omega)\}$ determines the ***spatial*** wavelength $\lambda(\omega)$, the ***phase*** speed $v_\phi(\omega)$ and also the ***group*** speed $v_g(\omega)$ of the monochromatic *EM* plane wave in the conductor:

$$\lambda(\omega) = \frac{2\pi}{k(\omega)} = \frac{2\pi}{\Re e\{\tilde{k}(\omega)\}}$$

$$v_\phi(\omega) \equiv \frac{\omega}{k(\omega)} = \frac{\omega}{\Re e\{\tilde{k}(\omega)\}}$$

$v_\phi(\omega)$ = propagation speed of a ***point*** on waveform that has ***constant phase*** Φ.

Phase $\Phi \equiv (kz - \omega t) =$ ***constant***. A constant phase ***point*** on the waveform moves: $z(t) = \Phi/k + v_\phi t$.

$$v_g(\omega) \equiv \frac{1}{dk(\omega)/d\omega} = \left[\frac{dk(\omega)}{d\omega}\right]^{-1}$$

$v_g(\omega)$ = propagation speed of ***energy*** / ***information***.

Assignment :

1. Find the wavelength and propagation speed in copper for radio wave at 1MHz compare the corresponding value in air.
2. Why high frequency wave propagate through the surface of the good conductor?
3. Explain the terms good and poor conductor depends on frequency.
4. Show that the skin depth in a good conductor is $(2/\sigma)\sqrt{(\frac{\epsilon}{\mu})}$.