Semester-IV

B.Sc (Honours) in Physics

C8T: Mathematical Physics III

Lecture

on

Eigen-values and Eigenvectors

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Lecture-IV

(Specially: Cayley- Hamiliton Theorem)

Cayley- Hamiliton Theorem.

Cayley–Hamilton Theorem. *A matrix satisfies its own characteristic equation. That is, if the characteristic equation of an $n \times n$ matrix $\mathbf{A}$ is $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$,* then

$$\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_1\mathbf{A} + a_0\mathbf{I} = \mathbf{0}.$$

Note once again that when we change a scalar equation to a matrix equation, the unity element 1 is replaced by the identity matrix $\mathbf{I}$.

Example 1 Verify the Cayley–Hamilton theorem for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution The characteristic equation for $\mathbf{A}$ is $\lambda^2 - 4\lambda - 5 = 0$.

$$\mathbf{A}^2 - 4\mathbf{A} - 5\mathbf{I} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 16 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8-0 \\ 16-16-0 & 17-12-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}.$$

Example 2 Verify the Cayley–Hamilton theorem for

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Solution The characteristic equation of $\mathbf{A}$ is $(3-\lambda)(-\lambda)(4-\lambda) = 0$.

$$(3\mathbf{I}-\mathbf{A})(-\mathbf{A})(4\mathbf{I}-\mathbf{A}) = \left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}\right)\left(-\begin{bmatrix} 3 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}\right)$$

$$\left(\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ -2 & 3 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -3 & 0 & 1 \\ -2 & 0 & -1 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -2 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ -2 & 3 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -3 & 0 & -3 \\ -2 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}.$$

Applications of Cayley–Hamilton theorem

Cayley–Hamilton theorem is a new method for finding the inverse of a nonsingular matrix. If

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

is the characteristic equation of a matrix **A**,
Then det(**A**) = $(-1)^n a_0$. Thus, **A** is invertible if and only if $a_0 \neq 0$.

Now assume that $a_0 \neq 0$. By the Cayley–Hamilton theorem, we have

$$\mathbf{A}^n + \mathbf{a}_{n-1}\mathbf{A}^{n-1} + \cdots + \mathbf{a}_1\mathbf{A} + \mathbf{a}_0\mathbf{I} = 0,$$

$$\mathbf{A}[\mathbf{A}^{n-1} + \mathbf{a}_{n-1}\mathbf{A}^{n-2} + \cdots + \mathbf{a}_1\mathbf{I}] = \mathbf{a}_0\mathbf{I}$$

Or , $\mathbf{A}[-1/\mathbf{a}_0 (\mathbf{A}^{n-1} + \mathbf{a}_{n-1}\mathbf{A}^{n-2} + \cdots + \mathbf{a}_1\mathbf{I})] = \mathbf{I}$

Thus inverse of A is,

$\mathbf{A}^{-1} = (-1/\mathbf{a}_0)(\mathbf{A}^{n-1} + \mathbf{a}_{n-1}\mathbf{A}^{n-2} + \cdots + \mathbf{a}_1\mathbf{I})$

Example 3 Using the Cayley–Hamilton theorem, find $\mathbf{A}^{-1}$ for

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix}.$$

Solution The characteristic equation for $\mathbf{A}$ is $\lambda^3 - 3\lambda^2 - 9\lambda + 3 = 0$. Thus, by the Cayley–Hamilton theorem,

$$\mathbf{A}^3 - 3\mathbf{A}^2 - 9\mathbf{A} + 3\mathbf{I} = \mathbf{0}.$$

Hence

$$\mathbf{A}^3 - 3\mathbf{A}^2 - 9\mathbf{A} = -3\mathbf{I},$$

$$\mathbf{A}(\mathbf{A}^2 - 3\mathbf{A} - 9\mathbf{I}) = -3\mathbf{I},$$

or,

$$\mathbf{A}\left(\frac{1}{3}\right)(-\mathbf{A}^2 + 3\mathbf{A} + 9\mathbf{I}) = \mathbf{I}.$$

Thus,

$$\mathbf{A}^{-1} = \left(\frac{1}{3}\right)\left(-\mathbf{A}^2 + 3\mathbf{A} + 9\mathbf{I}\right)$$

$$= \frac{1}{3}\left(\begin{bmatrix} -9 & 0 & -12 \\ -4 & -1 & -4 \\ -8 & 4 & -17 \end{bmatrix} + \begin{bmatrix} 3 & -6 & 12 \\ 0 & -3 & 6 \\ 6 & 0 & 9 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 3 & -6 & 0 \\ -4 & 5 & 2 \\ -2 & 4 & 1 \end{bmatrix}.$$

Problems

Verify the Cayley–Hamilton theorem and use it to find $\mathbf{A}^{-1}$, where possible, for:

1. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$,

2. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$,

3. $\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$,

4. $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}$,

5. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.