ELECTRIC DIPOLE

Electric dipole or bipole :—

Two equal and opposite ~~charges~~ point charges separated by a distance small compared to ~~the distance small compared to~~ the distance at which the potential and field are calculated, is called electric dipole.

Let the charges $+q$ and $-q$ separated by a small distance l, constitute an electric dipole.

$$-q \xrightarrow{\quad \vec{l} \quad} +q$$

Electric dipole moment :—

The product of magnitude of one charge and the distance between them is called electric dipole moment. Electric dipole moment is a vector quantity directed from negative charge to positive charge.

Electric dipole moment $\vec{P}$ or $\vec{M} = q\vec{l}$

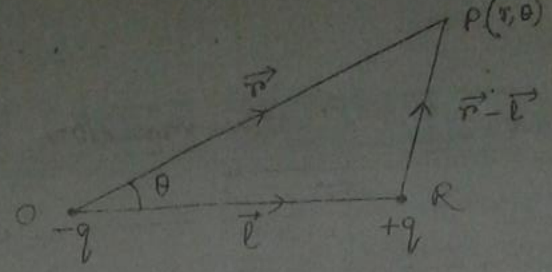
S.I. unit of electric dipole moment $\Rightarrow$ Coulomb – meter.

Obtain expression for electric potential and field due to an electric dipole at a distance very large from the origin compared to the length of the dipole.

Let us consider an electric dipole having charges $-q$ and $+q$ separated by a small distance $\vec{l}$.

Thus electric dipole moment $\vec{M} = q\vec{l}$

Let us consider, the co-ordinate system taking origin at the position of negative charge. ~~Let~~ We want to calculate the potential and field at $P(r, \theta)$.



Potential at P due to charge $-q = \dfrac{-q}{4\pi\epsilon_0 |\vec{r}|}$

Potential at P due to charge $+q = \dfrac{+q}{4\pi\epsilon_0 |\vec{r}-\vec{l}|}$

Total potential at P due to the dipole is,

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\left[\frac{-1}{|\vec{r}|} + \frac{1}{|\vec{r}-\vec{l}|}\right] \quad \text{——— (1)}$$

Now,
$$
\begin{aligned}
|\vec{r}-\vec{l}|^2 &= (\vec{r}-\vec{l})\cdot(\vec{r}-\vec{l}) \\
&= \vec{r}\cdot\vec{r} - \vec{r}\cdot\vec{l} - \vec{l}\cdot\vec{r} + \vec{l}\cdot\vec{l} \\
&= r^2 + l^2 - 2rl\cos\theta \\
&= r^2\left(1 + \frac{l^2}{r^2} - \frac{2l\cos\theta}{r}\right) \\
&= r^2\left(1 - \frac{2l\cos\theta}{r}\right)
\end{aligned}
\qquad \left\{ \frac{l}{r} = \text{small} \;\; \therefore \frac{l^2}{r^2} \to 0 \right.
$$

$$\Rightarrow |\vec{r}-\vec{l}| = r\left(1 - \frac{2l\cos\theta}{r}\right)^{1/2}$$

$$
\begin{aligned}
\Rightarrow \frac{1}{|\vec{r}-\vec{l}|} &= \frac{1}{r}\left(1 + \frac{l\cos\theta}{r}\right) \\
&= \frac{1}{r} + \frac{l\cos\theta}{r^2} \\
&= \frac{1}{r} + \frac{lr\cos\theta}{r^3} \\
&= \frac{1}{r} + \frac{\vec{l}\cdot\vec{r}}{r^3}
\end{aligned}
$$

From equation (1), we get ——

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r} + \frac{1}{r} + \frac{\vec{l}\cdot\vec{r}}{r^3}\right]$$

$$\boxed{V(r,\theta) = \frac{q\,\vec{l}\cdot\vec{r}}{4\pi\epsilon_0 r^3}} \quad \text{——— (1) A}$$

Since $q\vec{l} = \vec{M}$, dipole moment.

$$\boxed{V(r,\theta) = \frac{\vec{M}\cdot\vec{r}}{4\pi\epsilon_0 r^3}} \quad \text{———— (1) B}$$

Since $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

$$\therefore \boxed{V(r,\theta) = -\frac{\vec{M}\cdot\vec{\nabla}(1/r)}{4\pi\epsilon_0}} \quad \text{———— (1) C}$$

Since $\vec{M}\cdot\vec{r} = q\vec{l}\cdot\vec{r} = q\,l\,r\cos\theta$

$$\therefore \boxed{V(r,\theta) = \frac{q\,l\cos\theta}{4\pi\epsilon_0 r^2}} \quad \text{———— (1) D}$$

$$= \frac{M\cos\theta}{4\pi\epsilon_0 r^2}$$

Expression of electric field ————

Electric field $\vec{E} = -\vec{\nabla}V$

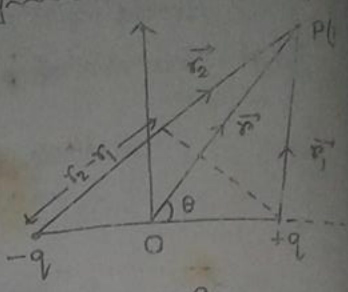
$$= -\left[\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}\right]\left(\frac{M\cos\theta}{4\pi\epsilon_0 r^2}\right)$$

$$= -\frac{M}{4\pi\epsilon_0}\left[-\frac{2\cos\theta}{r^3}\hat{r} - \frac{\sin\theta}{r^3}\hat{\theta}\right]$$

$$\boxed{\vec{E}(r,\theta) = \frac{M}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]} \quad \text{———— (2) A}$$

Potential and field due to an electric dipole at a point when the origin of the co-ordinate system is taken at the middle point of the dipole.

Let the distance of p from $-q$ charge $= \vec{r_2}$ and the distance from $+q$ charge $= \vec{r_1}$.



Then potential at p due to charge $-q = \dfrac{-q}{4\pi\epsilon_0 r_2}$

potential at p due to charge $+q = \dfrac{+q}{4\pi\epsilon_0 r_1}$

Thus potential at p due to dipole

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{r_2 - r_1}{r_1 r_2}\right)$$

From figure, $\left.\begin{array}{l} r_2 - r_1 = l\cos\theta \\ \text{and } r_1 r_2 = r^2 \end{array}\right\}$ since l is very small compared to r $(r_1 = r_2) = r$

$$\therefore V(r,\theta) = \frac{q}{4\pi\epsilon_0} \cdot \frac{l\cos\theta}{r^2}$$

$$= \frac{ql\cos\theta}{4\pi\epsilon_0 r^2}$$

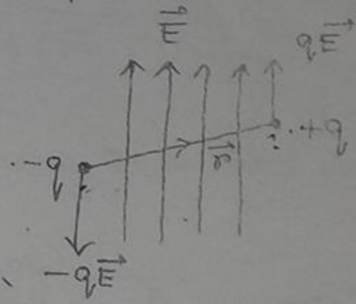
$$V(r,\theta) = \frac{M\cos\theta}{4\pi\epsilon_0 r^2}$$

or $$V(r,\theta) = \frac{\vec{M}\cdot\vec{r}}{4\pi\epsilon_0 r^3}$$

And $$\vec{E} = \frac{M}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$ as before

Torque acting on an electric dipole placed in external field :-

Let a dipole of moment $\vec{M}$, consisting of charges $+q$ and $-q$ separated by a distance $\vec{r}$ is placed in an external uniform electric field $\vec{E}$.



~~Let the~~

Force acting on charge $-q = -q\vec{E}$

Force acting on charge $+q = q\vec{E}$

These two equal and opposite forces constitute a couple.

The moment of this couple force is torque $\vec{\tau} = \vec{r} \times \vec{F}$

$$= \vec{r} \times q\vec{E}$$

$$= q\vec{r} \times \vec{E}$$

$$= \vec{M} \times \vec{E}$$

As $\vec{M} = q\vec{r}$

$$\therefore \boxed{\vec{\tau} = \vec{M} \times \vec{E}}$$

ELECTRICAL IMAGE

What is electrical image? V.U – 1993, 1996, 1998, 2000, 2002

Electrical image — An electrical image is defined as a point charge on one side of a surface, which will produce on other side of the surface the same electric field as is produced by the actual electrification of the surface.

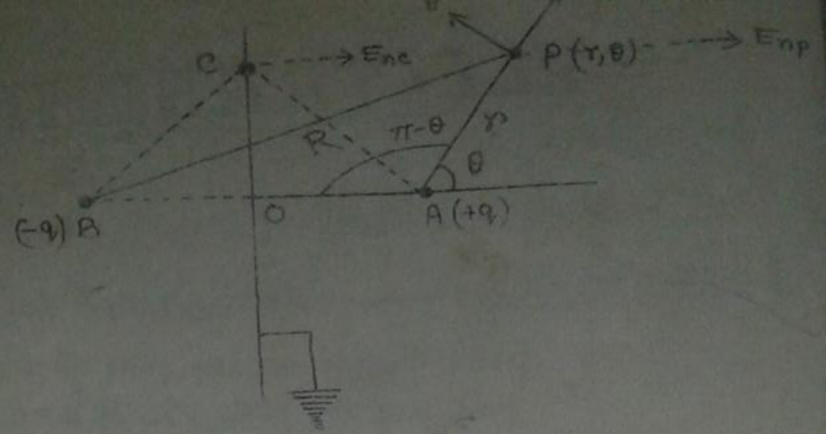
State its usefulness in solving electrostatic problems. V.U – 2000

By the method of electrical image, we can calculate —
i) Electric potential.
ii) Electric field.
iii) surface density of charge
iv) Total induced charge.

Moreover — a) The position of and magnitude of image charges are such that it does not involve any complicated diff. equation.
b) It does not require any knowledge of surface density of charge but it can give surface density of charge.

A point charge is placed in front of an infinite earthed conducting plane. Calculate — V.U – 1993, 1996, 2002
i) Electric potn. and field.
ii) Surface density of charge on the plane.
iii) Total induced charge on the plane.
iv) Force of attraction between the charges.

Let a point charge $+q$ be placed at a distance d from from an infinite earthed conducting plane such that the potn. of the conducting plane is zero.



We want to calculate the electric potn. and field at a Point P (r, θ). We take the origin of co-ordinate system at the position of point charge (A).

The conditions obeyed in this problem —

i) On the right hand side of the plane Laplace's equation holds everywhere except at a point A where the large is located

ii) The potential at everywhere on the conducting plane must be zero.

iii) The potential at infinity must be zero.

The symmetry of the problem suggests that the induced charge on the conducting plane may be replaced by an image charge "$-q$" on the left side of the conducting plane at a point B such that $OA = OB = d$.

Potential and field at P —

Thus the problem consists of two charges –

a) The real charge $+q$ at A at a distance d.

b) The image charge $-q$ at B at a distance d.

Hence potential at P,
$$\phi = \frac{q}{4\pi\epsilon_0 \cdot AP} + \frac{-q}{4\pi\epsilon_0 \cdot BP}$$
$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{AP} - \frac{1}{BP}\right]$$
$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{(r^2+4d^2+4rd\cos

As from figure, $\cos(\pi-\theta) = \dfrac{(AB)^2+(AP)^2-(BP)^2}{2\cdot AB\cdot AP}$

or, $(BP)^2 = 4d^2 + r^2 - 2\cdot 2d\cdot r\cdot \cos(\pi-\theta)$

or, $BP = (4d^2+r^2+4rd\cos\theta)^{1/2}$

Radial component of electric field at P,

$$E_r = -\frac{\partial\phi}{\partial r} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r^2} - \frac{r+2d\cos\theta}{(r^2+4d^2+4rd\cos\theta)^{3/2}}\right] \quad (2)$$

Transverse component of electric field at P,

$$E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = +\frac{1}{r}\frac{q}{4\pi\epsilon_0}\cdot\frac{4rd\sin\theta}{(r^2+4d^2+4rd\cos\theta)^{3/2}}$$

$$= \frac{q}{4\pi\epsilon_0}\cdot\frac{2d\sin\theta}{(r^2+4d^2+4rd\cos\theta)^{3/2}} \quad (3)$$

Electric field vector,

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r^2} - \frac{r+2d\cos\theta}{(r^2+4d^2+4rd\cos\theta)^{3/2}}\right]\hat{r}$$
$$+ \frac{q}{4\pi\epsilon_0}\cdot\frac{\boxed{2}d\sin\theta}{(r^2+4d^2+4rd\cos\theta)^{3/2}}\hat{\theta} \quad (4)$$

(1) Surface density of charges on the conducting plane :

Normal component of electric field at P,

$$E_{np} = E_r\cos\theta + E_\theta\cos(\pi/2+\theta)$$
$$= E_r\cos\theta - E_\theta\sin\theta$$
$$= \frac{q}{4\pi\epsilon_0}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta+2d\cos^2\theta+2d\sin^2\theta}{(r^2+4d^2+4rd\cos\theta)^{3/2}}\right]$$
$$= \frac{q}{4\pi\epsilon_0}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta+2d}{(r^2+4d^2+4rd\cos\theta)^{3/2}}\right]$$

Normal component of electric field at C, put $r = R$

$\cos(\pi - \theta) = d/R$

$\therefore \cos\theta = -d/R$

$$E_n = \frac{q}{4\pi\epsilon_0}\left[\frac{-d/R}{R^2} - \frac{R\cdot(-d/R) + 2d}{\{R^2 + 4d^2 + 4Rd(-d/R)\}^{3/2}}\right]$$

$$= -\frac{q}{4\pi\epsilon_0}\left[\frac{d}{R^3} + \frac{d}{(R^2+4d^2-4d^2)^{3/2}}\right]$$

$$= -\frac{q}{4\pi\epsilon_0}\cdot\frac{2d}{R^3} \qquad \text{(5)}$$

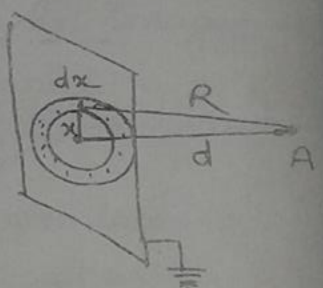
Hence surface density of charge at the plane (at C)

$$\sigma = \epsilon_0 E_n \qquad \text{as } E_n = \sigma/\epsilon_0$$

$$\therefore \boxed{\sigma = -\frac{qd}{2\pi R^3}}$$

(7) Total induced charge on the plane —

Let us take an elementary ring of radius x and thickness dx on the plane.



Where $x^2 = R^2 - d^2$

or, $x\,dx = R\,dR$

Charge induced on the elementary ring

$$dQ = 2\pi x\,dx\cdot\sigma$$

$$= 2\pi R\,dR\cdot\left(-\frac{qd}{2\pi R^3}\right)$$

$$= -qd\cdot\frac{dR}{R^2}.$$

$\therefore$ Total charge induced on the plane,

$$Q = -qd\int_d^{\infty}\frac{dR}{R^2}$$

$$\therefore \boxed{Q = -q}$$

Force ~~on~~ between the charges

The problem consists of two charges —

a) a real charge $+q$ at A at a distance d

b) an image charge $-q$ at B at a distance d

Hence force between the charges,

$$F = \frac{q \cdot (-q)}{4\pi\epsilon_0 \cdot (2d)^2}$$

or,

$$\boxed{F = -\frac{q^2}{4\pi\epsilon_0 \times 4d^2}}$$

attractive