17 TUESDAY ① 2020 March

12th Week • 077-289

Electrostatic field :—

Electrostatic field is the space surrounding a point charge at every point of which the influence (attraction or repulsion) is experienced.

Intensity of electric field or electric field :—

The force experienced by a point unit positive charge placed at any point in the field of a charge is called intensity of electric field at that point.

Mathematically $\vec{E} = \frac{q}{4\pi\epsilon_0 r^3}\vec{r}$ in free space

Remember: $\vec{E} = \frac{\vec{F}}{q}$

S.I unit of $\vec{E}$ ⇒ Volt/meter or N/C

Defination of electric field Lines :—

An electric field line is an imaginary line or curve drawn through a region of empty space, so that its tangent at any point is in the direction of the electric field vector at the point. The relative closeness of the lines at some place gives an idea about the intensity of electric field at that point.

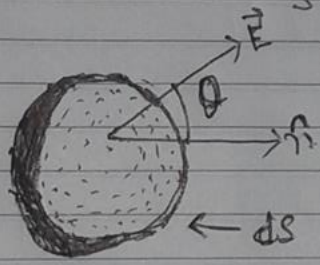
2020 MARCH

Wk	M	T	W	T	F	S	S
9	30	31					1
10	2	3	4	5	6	7	8
11	9	10	11	12	13	14	15
12	16	17	18	19	20	21	22
13	23	24	25	26	27	28	29

Flux of electric field :—

Consider a surface element $d\vec{S} = \hat{n}\,dS$ in an electric field $\vec{E}$, where $\hat{n}$ is the outward unit vector normal to the surface element. The quantity $d\phi = \vec{E}\cdot d\vec{S} = E\cos\theta\, dS$ is called the flux of $\vec{E}$ through $d\vec{S}$. If we consider that no/to numbers of lines of force emanate isotropically from a point charge 'q' then the idea of electric flux becomes meaningful. If we use this picture of electric field then $d\phi = \vec{E}\cdot d\vec{S}$ becomes equal to the number of lines of forces passing through the area $d\vec{S}$. The flux of $\vec{E}$ over any arbitary surface S is given by the integral,

$$\phi = \int d\phi = \int_S \vec{E}\cdot d\vec{S}$$



2020 APRIL

Wk	M	T	W	T	F	S	S
14			1	2	3	4	5
15	6	7	8	9	10	11	12
16	13	14	15	16	17	18	19
17	20	21	22	23	24	25	26
18	27	28	29	30			

Show that the electrostatic field of a point charge has zero curl or conservative in nature.

V.U - 2002

Solution —— The electrostatic field of a point charge q at a distance r is given by $\vec{E} = \frac{q}{4\pi\epsilon_0 r^3}\vec{r}$

$$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left[\frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\vec{\nabla}\left(\frac{1}{r^3}\right)\times\vec{r} + \frac{1}{r^3}\vec{\nabla}\times\vec{r}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[-\frac{3}{r^5}(\vec{r}\times\vec{r}) + \frac{1}{r^3}(\vec{\nabla}\times\vec{r})\right]$$

$$= 0 \quad \text{as} \quad \vec{r}\times\vec{r} = 0, \quad \vec{\nabla}\times\vec{r} = 0$$

$\therefore \vec{r}\times\vec{r} = 0$

Check which of the following functions may be possible electrostatic field —— (i) $\vec{E} = (4y\hat{i} - 2x\hat{j} - \hat{k})A$

(ii) $\vec{E} = (2x\hat{i} - yz^2\hat{j} - \hat{k} - y^2z\hat{k})A$

Where A is a constant

C.U - 1998, V.U - 2009

Solution —— i) $\vec{E} = A(4y\hat{i} - 2x\hat{j} - \hat{k})$

$$\therefore \vec{\nabla}\times\vec{E} = A\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & -2x & -1 \end{vmatrix}$$

$$= A\left[\hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(-2-4)\right]$$

$$= -6A\hat{k}$$

Since $\vec{\nabla}\times\vec{E} \neq 0$, thus $\vec{E}$ is not electrostatic field here.

(ii) $\vec{E} = [2x\hat{i} - yz^2\hat{j} - (1+y^2z)\hat{k}]A$

$$\vec{\nabla}\times\vec{E} = A\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -yz^2 & -(1+y^2z) \end{vmatrix}$$

$$= A[\hat{i}(-2yz+2yz) + \hat{j}(0-0) + \hat{k}(0-0)]$$

$$= 0$$

Since $\vec{\nabla}\times\vec{E} = 0$, Thus here $\vec{E}$ represents electric field

Relation between electric field and potential

$$\vec{\nabla}\times\vec{E} = 0$$

Again $\vec{\nabla}\times(-\vec{\nabla}V) = 0$

Compairing we get $\boxed{\vec{E} = -\vec{\nabla}V}$

Taking dot product with $d\vec{r}$,

$$\text{i.e. } \vec{E}\cdot d\vec{r} = -\vec{\nabla}V\cdot d\vec{r}$$

$$\Rightarrow \vec{E}\cdot d\vec{r} = -dV$$

$$\Rightarrow \boxed{E = -\frac{dV}{dr}}$$

Potential due to a point charge

Electric field at a distance r from a point charge q.

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^3}\vec{r}$$

$$\therefore dV = -\vec{E}\cdot d\vec{r} = -\frac{q\,r\,dr}{4\pi\epsilon_0 r^3}$$

$$\int_0^V dV = -\frac{q}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{dr}{r^2}$$

$$\therefore \boxed{V = \frac{q}{4\pi\epsilon_0 r}}$$

Problem V.U-2005 (1)

An electric field $\vec{E} = y\hat{i} + x\hat{j}$ exists in a region. Taking the electrostatic potential at origin as zero, find the potential at the position P(a,a)

Solution: We know the relation between electrostatic field and potential as,

$$\vec{E} = -\vec{\nabla}V$$

$$\Rightarrow \vec{E} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) \quad \text{in x-y}$$

$$\therefore -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j}\right) = y\hat{i} + x\hat{j}$$

Thus, $\frac{\partial V}{\partial x} = -y$ and $\frac{\partial V}{\partial y} = -x$

Int. $V = -xy + c_1$ Int: $V = -xy + c_2$

At the origin (0,0), potential (V) is zero.

i.e When $x=0$, $y=0$ then $V=0$ $\therefore c_1 = c_2 = 0$

$$\therefore \boxed{V = -xy}$$

∴ Potential at (a,a), $V = -a^2$ Ans:

V.V. 2008

State Gauss's theorem in electrostatics — V.U-'95,99,'01,'03

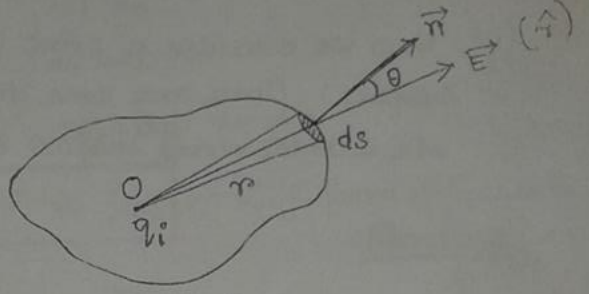
Statement — Gauss's theorem in electrostatics states that flux of electric intensity across a closed surface in free space is equal to $\frac{1}{\epsilon_0}$ times the charge enclosed by the surface or zero if the charge is outside the surface.

Mathematically, $\oint \vec{E} \cdot \hat{n}\, ds = \frac{q}{\epsilon_0}$ When the charge inside the closed surface

$= 0$ When the charge outside the closed surface.

Derivation —

(i) Let us consider a point charge q_i at O inside a closed surface.



Electric field at ds at a distance r from O

$$\vec{E} = \frac{q_i}{4\pi\epsilon_0 r^3}\vec{r} \quad V/m$$

Flux of electric field due to charge q_i through the elementary area ds,

$$d\phi = \vec{E}\cdot\hat{n}\, ds = \frac{q_i}{4\pi\epsilon_0 r^3}\vec{r}\cdot\hat{n}\, ds$$

$$= \frac{q_i}{4\pi\epsilon_0 r^2} ds\cos\theta$$

$$= \frac{q_i}{4\pi\epsilon_0} d\omega$$

Where $d\omega = \frac{ds\cos\theta}{r^2}$, is the solid angle subtended by ds at O.

Thus total flux through the closed surface is,

$$\oint \vec{E}.\hat{n}\,ds = \frac{q_i}{4\pi\epsilon_0}\oint d\omega$$

$$= \frac{q_i}{4\pi\epsilon_0} \times 4\pi$$

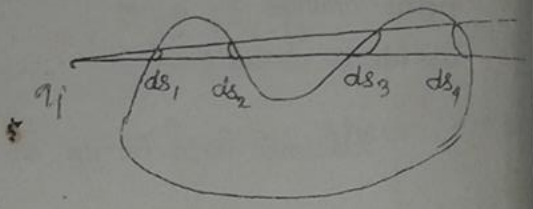
$$= \frac{q_i}{\epsilon_0}$$

If there are many point charges $q_i, q_j, q_k \cdots$ inside the closed surface,

then $$\oint \vec{E}.\hat{n}\,ds = \frac{q_i + q_j + q_k + \cdots}{\epsilon_0}$$

$$\boxed{\oint \vec{E}.\hat{n}\,ds = \frac{q}{\epsilon_0}}$$

(ii) Now we consider a point charge q_i is outside the closed surface. Then any cone drawn from O (any external pt.) will cut the closed surface at even number of times.



Flux of $\vec{E}$ through $ds_1 = -\frac{q_i}{4\pi\epsilon_0}\frac{ds_1\cos\theta_1}{r_1^2} = -\frac{q_i}{4\pi\epsilon_0}d\omega$

Flux of $\vec{E}$ through $ds_2 = \frac{q_i}{4\pi\epsilon_0}\frac{ds_2\cos\theta_2}{r_2^2} = \frac{q_i}{4\pi\epsilon_0}d\omega$

Hence $$\boxed{\oint \vec{E}.\hat{n}\,ds = 0}$$

Q. Express Gauss's theorem in differential form.
V.U - 1999, 2001

According to Gauss's theorem in electrostatics,

$$\oint \vec{E} \cdot \hat{n}\, ds = \frac{q}{\epsilon_0}$$

$$\Rightarrow \oint \vec{E} \cdot \hat{n}\, ds = \frac{1}{\epsilon_0} \iiint \rho\, dv \quad \text{—①}$$

Where ρ is the volume density of charge and dv is arbitrary volume.

According to Gauss's divergence theorem, in eqn. ①,

$$\iiint \vec{\nabla} \cdot \vec{E}\, dv = \iiint \frac{\rho}{\epsilon_0} dv$$

$$\Rightarrow \iiint \left(\vec{\nabla} \cdot \vec{E} - \frac{\rho}{\epsilon_0}\right) dv = 0$$

Since volume taken is arbitrary $dv \neq 0$

Thus $\boxed{\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0}$ — Diff. form of Gauss's theorem

Q. What are the premises on which Gauss's theorem based on ?

Q. Does the theorem differ from Coulomb's law ? V.U - 1999
V.U - 2006

① Gauss's theorem based on —
i) the inverse square nature of the interaction
and ii) the principle of superposition.

② Gauss's theorem is a necessary consequence of the inverse square law thus ie Coulomb's law.

When we do apply Gauss's law ?

The necessary condition for applying Gauss's law is uniform, symmetrical charge distribution and symmetrical

charge distribution.

Gaussian surface.

Example as — i) Spherical symmetry (sphere)
ii) Cylindrical symmetry (cylinder)
iii) Plane symmetry (pill box)

What is Poisson's ~~Laplace's~~ equation in electrostatics? V.U - 2002, 2012

Poisson's equation and Laplace's equation

① Poisson's equation —

If V be the electrostatic potential, then $\vec{E} = -$

From Gauss's law in electrostatics, $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$

$\Rightarrow \vec{\nabla} \cdot (-\vec{\nabla} V) = \rho/\epsilon_0$

$\Rightarrow \boxed{\nabla^2 V = -\rho/\epsilon_0}$

② Laplace's equation —

If there is no charge within the surface, then $\rho = 0$

Thus from Poisson's equation, $\boxed{\nabla^2 V = 0}$

Importance of Poisson's and Laplace's equation

Once the Poisson's equation is solved, we get the scalar potential V. Hence electric field $\vec{E}$ is calculated from the relation $\vec{E} = -\vec{\nabla} V$. Now when $\vec{E}$ and $\vec{V}$ are kno all rest properties can be understood.

In charge free region, the solution of Laplace's equ gives the potential (V) there, under certain boundary conditions.

Poisson's and Laplace's equation

Properties and problems on Laplace's equation and its solution

Laplace's equation, $\nabla^2\phi = 0$

Significance of Laplace's equation —

Laplace's equation is an important differential equation. A function which satisfies Laplace's equation is known as — Spherical harmonic. A very important property of such function is —

The average value of a function $\phi(x,y,z)$ satisfying Laplace's equation over any spherical surface is equal to its value at the centre of spherical surface.

Uniqueness theorem :— Two solutions of Laplace's equation obeying the same boundary conditions differ at best by a constant.

Proof — Let us assume ϕ_1 and ϕ_2 are two solutions of Lapl equation in a volume V composed of surfaces S_1, S_2

Let ϕ_1 and ϕ_2 satisfy the same boundary conditions, i.e.,

either i) Dirichlet condition ; $\phi_1 = \phi_2$ on the bounding sur

or ii) Neumann condition ; $\frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n}$ on the boundin

Let $\phi = \phi_1 - \phi_2$

or, $\nabla^2\phi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0$ Since $\nabla^2\phi_1 = \nabla^2\phi_2 = 0$

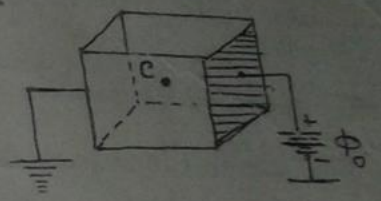
or $\nabla^2\phi = 0$ ——— (1)

Again, according to Gauss's divergence Theorem,

$$\int_V \vec{\nabla} \cdot (\phi \vec{\nabla}\phi)\, dV = \int_{S_1+S_2+\cdots S_n} \phi \vec{\nabla}\phi \cdot \hat{n}\, ds$$

remaining side is insul
is maintained at a potenti
static potential at the centr

Solution —



The elec
of the c

Since sin
w.r.to c

$$\therefore \phi_c = C_0 \sum_{n=1}^{n} \phi_n$$

If all the sides have same

$$\therefore C_0 = \frac{1}{6}$$

If one side the potential of o
earthed then $\boxed{\phi_c = \frac{\phi_0}{6}}$

or, $\int_V (\phi \nabla^2\phi + \vec{\nabla}\phi \cdot \vec{\nabla}\phi)\, dV = 0$ as $\vec{\nabla}\phi \cdot \hat{n} = 0$ according to Neumann condition

or, $\int_V |\vec{\nabla}\phi|^2 dV = 0$ as $\nabla^2\phi = 0$

Since integral of $|\vec{\nabla}\phi|^2$ vanishes, $\vec{\nabla}\phi$ must be zero at each point in V i.e $\vec{\nabla}\phi = 0$

$\therefore \phi = \text{const.}$

$\therefore \underline{\phi_1 - \phi_2 = c}$ Proved

Problem (2) Starting from the expression of the electrostatic potential, $\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}_0)\,dV_0}{|\vec{r}-\vec{r}_0|}$

obtain Poisson's equation : $\nabla^2\phi = -\rho/\epsilon_0$

Where symbols have their usual meanings.

Solution —— If source point and observation point are very close to each other then $\rho(\vec{r}_0) \approx \rho(\vec{r}) = \rho$

Then $\vec{\nabla}\phi = -\frac{1}{4\pi\epsilon_0}\int \frac{\rho\, dV_0\,(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}$

$$= -\frac{\rho}{4\pi\epsilon_0}\int \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}\,dV_0$$

$$\therefore \nabla^2\phi = \vec{\nabla}\cdot\vec{\nabla}\phi = -\frac{\rho}{4\pi\epsilon_0}\int \vec{\nabla}\cdot\frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}\,dV_0$$

$$= -\frac{\rho}{4\pi\epsilon_0}\oint_S \frac{(\vec{r}-\vec{r}_0)\cdot\hat{n}}{|\vec{r}-\vec{r}_0|^3}\,ds_0$$ due to Gauss

S being the surface bounding the

Again solid angle subtended by the closed surface is

$$\therefore \nabla^2\phi = -\frac{\rho}{4\pi\epsilon_0}\times 4\pi$$

$$\therefore \boxed{\nabla^2\phi = -\rho/\epsilon_0}$$ which is Poisson's equation

Problem (4) The potential in a medium is given by $\phi(r) = \frac{q}{4\pi\epsilon_0}$. Obtain the corresponding electric field and charge density.

P.U - 94
P.U - 2000
V.U - 2008

008

Solution — $\phi = \frac{q}{4\pi\epsilon_0} \frac{e^{-r/\lambda}}{r}$

Electric field $\vec{E} = -\vec{\nabla}\phi$

$$= -\frac{d\phi}{dr}\hat{r}$$

$$= -\frac{d}{dr}\left[\frac{q}{4\pi\epsilon_0}\frac{e^{-r/\lambda}}{r}\right]\hat{r}$$

$$= -\frac{q}{4\pi\epsilon_0}\left[-\frac{1}{\lambda}\frac{e^{-r/\lambda}}{r} - \frac{e^{-r/\lambda}}{r^2}\right]\hat{r}$$

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0}\hat{r}\, e^{-r/\lambda}\left[\frac{1}{\lambda r} + \frac{1}{r^2}\right]}$$

To calculate charge density, we apply Poisson's equation

$$\nabla^2\phi = -\rho/\epsilon_0$$

or $\rho = -\epsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \cdots\right]$

$$= -\epsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(-r^2\vec{E}\right)\right] \quad \text{or} \quad \vec{E} = -\frac{\partial\phi}{\partial r}$$

$$= \epsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left\{\frac{q}{4\pi\epsilon_0}\left(\frac{r^2 e^{-r/\lambda}}{\lambda r} + \frac{r^2 e^{-r/\lambda}}{r^2}\right)\right\}\right]$$

$$= \frac{\epsilon_0}{r^2} \times \frac{q}{4\pi\epsilon_0}\cdot\frac{\partial}{\partial r}\left[\frac{r e^{-r/\lambda}}{\lambda} + e^{-r/\lambda}\right]$$

$$= \frac{q\epsilon_0}{4\pi\epsilon_0 r^2}\left[\frac{-r e^{-r/\lambda}}{\lambda\times\lambda} + \frac{e^{-r/\lambda}}{\lambda} - \frac{1}{\lambda}e^{-r/\lambda}\right]$$

$$= -\frac{q\epsilon_0 e^{-r/\lambda} r}{4\pi\epsilon_0\lambda^2 r^2}\left[\text{terms}\right]$$

$$= -\frac{q e^{-r/\lambda}}{4\pi\lambda^2 r}.$$

Question ⑤ V.U-2009

Write down Laplace's equation and show that the potential function $V = x^2 + y^2 - 2z^2$ satisfy Laplace's equation.

Laplace's equation in charge free region, $\nabla^2 V = 0$

Where V is potential function.

In 3-D cartesian co-ordinates (x, y, z), Laplace's equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{——} ①$$

Second part: Given, $V = x^2 + y^2 - 2z^2$

$$\therefore \frac{\partial^2 V}{\partial x^2} = 2.$$

Similarly, $\frac{\partial^2 V}{\partial y^2} = 2$

and $\frac{\partial^2 V}{\partial z^2} = -4.$

Substituting these values in equation ①, we get.

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 2 + 2 - 4 = 0$$

$$\Rightarrow \boxed{\nabla^2 V = 0}$$ Which is Laplace's eqn.

Hence the above potential function satisfy Laplace's equation.