

- SEMESTER –II (HONOURS)
- PAPER : CC4T
- TOPIC: ENERGY OF VIBRATING STRING

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Sub topics : struck string ,energy of string .

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Energy of vibrating string:

Let the displacement of the string be

$$\psi(x, t) = A \cos(kx - \omega t). \quad (1)$$

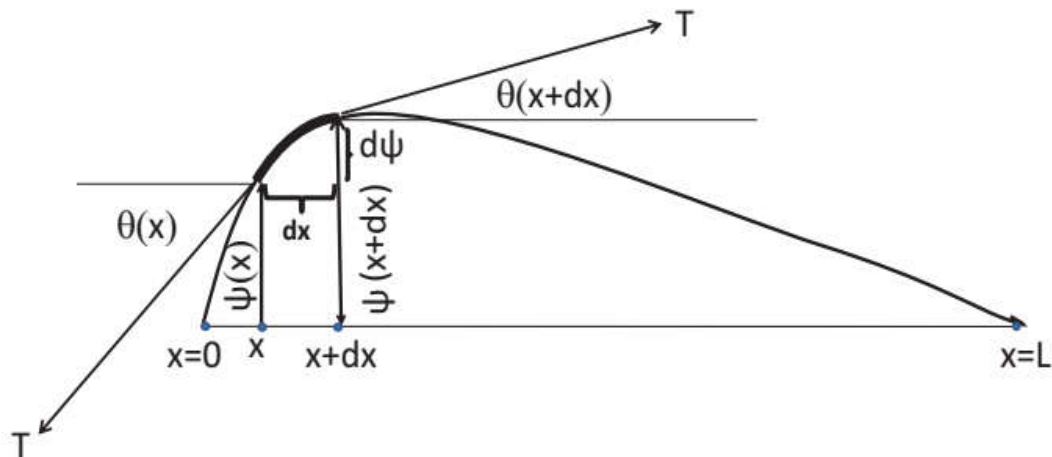


Figure 1: The string under tension T , has mass μ per unit length and is fixed at $x = 0$ and $x = L$. The highlighted segment has a width dx , with the same tension T pulling the two ends but at slightly different angles. The displacement ψ and angles θ are exaggerated for clarity. The derivation is valid only when all these are very small.

Now, if it's an infinitely long string, the energy in it is infinite, so you define the energy per unit length. Take a portion of the string, associated with a segment of width dx , and ask, "How much energy does it have?"

The energy has a kinetic part and a potential part. The kinetic part is simple:

$$dK = \frac{1}{2} \mu dx \left[\frac{\partial \psi}{\partial t} \right]^2. \quad (2)$$

This is just the mass of the segment times the square of the velocity. For the solution in Eqn. 1 this becomes

$$dK = \frac{1}{2} \mu dx A^2 \omega^2 \sin^2(kx - \omega t). \quad (3)$$

The potential energy dU has a more subtle origin. The string is under tension T . When the string is displaced from equilibrium, a segment associated with the interval dx has a

length (see Figure 1)

$$dl = \sqrt{dx^2 + (d\psi)^2} = dx \sqrt{1 + \left[\frac{\partial\psi}{\partial x}\right]^2} \simeq dx \left(1 + \frac{1}{2} \left[\frac{\partial\psi}{\partial x}\right]^2\right). \quad (4)$$

Thus it has expanded by an amount

$$\delta l = dl - dx = \frac{1}{2} \left[\frac{\partial\psi}{\partial x}\right]^2 dx \quad (5)$$

against a tension T . So the work pumped in and stored as potential energy is

$$dU = T\delta l = \frac{1}{2}T \left[\frac{\partial\psi}{\partial x}\right]^2 dx. \quad (6)$$

For the solution in Eqn. 1 this becomes

$$dU = \frac{1}{2}Tk^2A^2 \sin^2(kx - \omega t) \cdot dx = \frac{1}{2}\mu A^2\omega^2 \sin^2(kx - \omega t) \cdot dx = dK \quad (7)$$

upon using

$$Tk^2 = T\frac{\omega^2}{v^2} = T\omega^2\frac{\mu}{T} = \mu\omega^2. \quad (8)$$

Notice that the kinetic and potential energies are equal at every value of x and t . In particular when the string bit is maximally displaced, it has no velocity and hence no dK . It also has no dU because the slope is zero and so is the stretch δl . When the displacement is zero, it has the greatest velocity and slope and hence the greatest dK and dU . This is unlike the harmonic oscillator, where due to the conservation of energy, K and U add up to a constant, and when one is large the other is small. This is not a problem for the string bit because it is not an isolated system, and energy is pumped into (or out of) it by its neighbors.

The total energy is

$$dE = dK + dU = 2dK = \mu dx A^2 \omega^2 \sin^2(kx - \omega t), \quad (9)$$

The energy per unit length or *energy density* u is then

$$u(x, t) = \mu A^2 \omega^2 \sin^2(kx - \omega t) \quad (10)$$

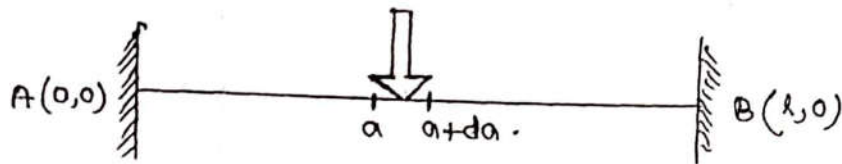
Of interest is the *average energy density* \bar{u} , where the average is over a full cycle in space (a wavelength) or a full cycle in time (a full period T). The average of \sin^2 is $\frac{1}{2}$ in either case and we have the final result

$$\bar{u} = \frac{1}{2}\mu A^2 \omega^2. \quad (11)$$

Struck string:

When the string is fixed at its ends and the string is struck by a hammer at a point, then the impulse imparts a velocity at the point of contact and all other points are rest, then initially, then the string is called struck string.

Example — piano.



Boundary conditions —

i) $y(x,0) = 0$

ii) $\left. \begin{aligned} \frac{dy}{dt} \Big|_{t=0} &= u ; a \leq x \leq a+da \\ &= 0 ; \text{other regions} \end{aligned} \right\} = \dot{y}_0 \text{ (say)}$

General equation of string,

$$y(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi vt}{l} + B_n \sin \frac{n\pi vt}{l} \right) \sin \frac{n\pi x}{l} \quad \text{--- (1)}$$

$$\therefore \frac{dy}{dt} = \sum_{n=1}^{\infty} \frac{n\pi v}{l} \left(-A_n \sin \frac{n\pi vt}{l} + B_n \cos \frac{n\pi vt}{l} \right) \sin \frac{n\pi x}{l} \quad \text{--- (2)}$$

Putting b.c. i) in equation (1), $A_n = 0$ as $\sin \frac{n\pi x}{l} \neq 0$.

Hence equation (1) reduces to,

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi vt}{l} \cdot \sin \frac{n\pi x}{l} \quad \text{--- (3)}$$

Putting b.c. ii) and $A_n = 0$ in equation (2),

$$\dot{y}_0 = \sum_{n=1}^{\infty} \frac{n\pi v}{l} \cdot B_n \cdot \sin \frac{n\pi x}{l} \quad \text{--- (4)}$$

To evaluate B_n , multiply both sides by $\sin \frac{n\pi x}{l}$ and integrate w.r.t. to x from $x=0$ to $x=l$.

$$i. \int_0^L y_0 \sin \frac{n\pi x}{L} dx = \sum_{n=1}^{\infty} \frac{n\pi v}{L} B_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L} dx$$

|| $\frac{L}{2}$ when $n=m$

We get only one terms when $n=m$,

$$\therefore B_n = \frac{2}{n\pi v} \int_0^L y_0 \sin \frac{n\pi x}{L} dx$$

$$= \frac{2}{n\pi v} \int_a^{a+da} u \sin \frac{n\pi x}{L} dx$$

Since da is very small quantity, the sin function may be assumed to a constant i.e. $\sin \frac{n\pi a}{L}$.

$$= \frac{2}{n\pi v} \sin \frac{n\pi a}{L} \int_a^{a+da} u dx$$

$$= \frac{2}{n\pi v} \cdot \sin \frac{n\pi a}{L} \cdot \langle u \rangle da \quad \text{as } \langle f(x) \rangle = \frac{1}{b-a} \int_a^b f(x) dx$$

$$\therefore \langle u \rangle = \frac{1}{da} \int_a^{a+da} u dx$$

Resultant wave equation,

$$y(x,t) = \frac{2\langle u \rangle da}{n\pi v} \cdot \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi a}{L} \sin \frac{n\pi x}{L} \cdot \sin \frac{n\pi vt}{L}$$

Young - Helmholtz law:

In this case Young Helmholtz law same as plucked string

Assignment :

1. state Young Helmholtz law of vibrating string . assuming the displacement equation of a struck string show that the law is valid for such string .
2. Define group velocity and phase velocity.
3. The dispersion relation for transverse wave propagating in a medium is given by $\omega^2 = \omega_p^2 + k^2 c^2$. show that : $v_g v_p = c^2$.
4. A three meter long wire of mass 0.5gm/m is stretch with tension of 20N. deduce the time taken by the transverse disturbance to travel across the length of the wire.