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Sem - II
Topic - Magnetic field

Magnetic field :

Magnetostatic force $\vec{F} = q(\vec{v} \times \vec{B})$:

or $|\vec{F}| = qvB$

or $B = \frac{F}{qv}$

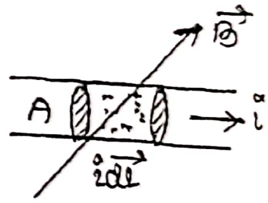
Therefore, magnetic field $\vec{B}$ is numerically equal to the force per unit charge ~~in~~ moving with unit velocity perpendicular to the field.

- ($\vec{B}$) is called magnetic field or magnetic flux density or magnetic induction.
- Unit of $\vec{B} \equiv$ Newton-second / coulomb-meter or $V \cdot s/m^2$.
  Also, $Weber/m^2$ or Tesla (T)
- Dimension of $\vec{B} \equiv MT^{-2}I^{-1}$

Magnetic force on a current carrying conductor :

Suppose a conducting wire, carrying a current i, is placed in a magnetic field $\vec{B}$.

Consider a small current element $i\vec{dl}$ of the wire, where the charge carriers move with a velocity $\vec{v}$ along the wire.



Force on each carrier $= q(\vec{v} \times \vec{B})$

If n be the concentration of charge carriers.

A be the cross-section of the element.

Then number of charge carriers present within the element $= n\vec{A} \cdot \vec{dl}$.

$\therefore$ Magnetic force on the element dl,

$$d\vec{F} = \boxed{n\vec{A} \cdot \vec{dl}}\, q(\vec{v} \times \vec{B})$$

$$= nq\vec{v} \cdot \vec{A}\, (\vec{dl} \times \vec{B}) \quad \text{as } \vec{v} \parallel \vec{dl}$$

$$= (\vec{J} \cdot \vec{A})\, \vec{dl} \times \vec{B}$$

$$= i\,\vec{dl} \times \vec{B}$$

$\therefore$ Magnetic force on a whole conductor ; $\boxed{\vec{F} = \int i\,\vec{dl} \times \vec{B}}$

Note: (1) When a closed current carrying loop is placed in a magnetic field, force

$$\vec{F} = \oint i\vec{dl} \times \vec{B}$$
$$= i(\oint \vec{dl}) \times \vec{B}$$
$$= 0 \quad \text{as } \oint \vec{dl} = 0$$

(2) Defination of magnetic field $\vec{B}$;

$$d\vec{F} = i\vec{dl} \times \vec{B}$$

or, $|\vec{B}| = \frac{|d\vec{F}|}{idl}$ i.e magnetic field ($\vec{B}$) is numerically equal to force per unit current element placed perpendicular to the magnetic field.

Origin of $\vec{B}$:

The electric field $\vec{E}$ satisfies the relation $\vec{\nabla}\cdot\vec{E} = +\rho/\epsilon_0$

where ρ is the electric charge density

In analogy, in magnetic case we may write,

$$\vec{\nabla}\cdot\vec{B} = K_m \rho_m$$

where $K_m \to$ mag. constant
$\rho_m \Rightarrow$ magnetic charge density.

But free magnetic poles do not exist, as magnetic poles are of equal strength and opposite, also occour in pairs. Therefore $\rho_m = 0$

$\therefore$ $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$, i.e $\vec{B}$ is always solenoidal.

Integratting above equation over a volume V, we obtain

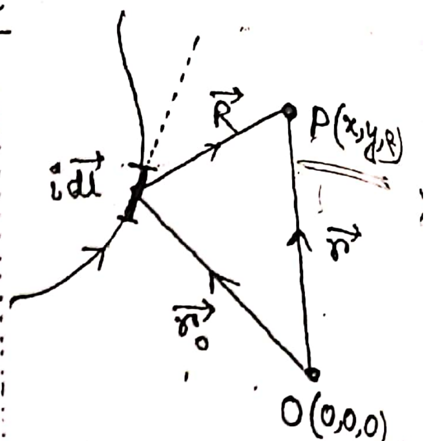
$$\iiint \vec{\nabla}\cdot\vec{B} = 0$$

or $\oint \vec{B}\cdot\hat{n}\, ds = 0$

or $\boxed{\phi = 0}$ Which shows that, magnetic lines of force are continuous, i.e there is no magnetic charge on which magnetic lines of force originate or terminate.

Bio-Savart law :

In case of line current element



$\vec{R} = \vec{r} - \vec{r_0}$

Magnetic flux density at any arbitary point p due to current element $i\vec{dl}$,

$$dB \propto i\,dl$$
$$\propto \sin\theta$$
$$\propto \frac{1}{R^2}$$

i.e $dB = \frac{\mu_0}{4\pi} \cdot \frac{i\,dl \sin\theta}{R^2}$

Which is directed perpendicular to the plane containing dl and R

Therefore vectorically it can be written as,

$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{i\vec{dl} \times \vec{R}}{R^3}}$$

Where $\mu_0 \Rightarrow 4\pi \times 10^{-7}$ N/A^2 called permeability in free space and $\vec{R} = \vec{r} - \vec{r_0}$.

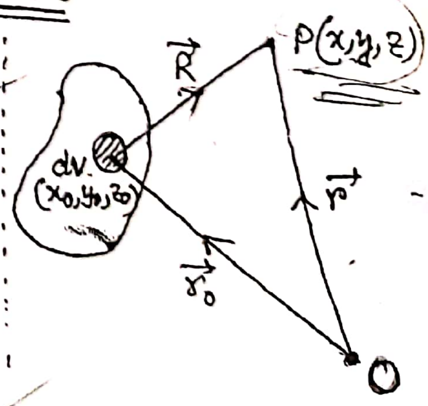
According to principle of superposition, total magnetic field $\vec{B}$ due to the entire line current.

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \int \frac{i\vec{dl} \times \vec{R}}{R^3}}$$

In case of volume current element

$i\vec{dl} = (\vec{J} \cdot \hat{n}\, ds)\, \vec{dl} = (\vec{dl} \cdot \hat{n}\, ds)\vec{J} = \vec{J}\, dv(\vec{r_0})$

Where $\vec{J}(\vec{r_0})$ is the current density vector.



So replacing line current element $i\vec{dl}$ by volume current element $\vec{J}dv$, we get-

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{J}\, dv \times \vec{R}}{R^3}$$

or $$\boxed{d\vec{B} = \frac{\mu_0}{4\pi} \frac{\vec{J} \times \vec{R}}{R^3}\, dv}$$

Where $\vec{J} \Rightarrow \vec{J}(\vec{r_0})$ or $\vec{J}(x_0,$
$\vec{R} = \vec{r} - \vec{r_0}$
$dv = dv(x_0, y_0, z_0)$

According to principle of superposition, total magnetic field $\vec{B}$ due to the entire volume distribution of current,

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\times\vec{R}}{R^3}\,dV}$$

Again $\vec{\nabla}\left(\frac{1}{R}\right) = -\frac{\vec{R}}{R^3}$, therefore another form -

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int_V \vec{\nabla}\left(\frac{1}{R}\right)\times\vec{J}\,dV}$$

Divergence of magnetic field :

According to Bio-Savart law, the magnetic field $\vec{B}$ at any point $P(x,y,z)$ due to a volume distribution of current $\vec{J}(\vec{r_0})$ is given by,

$$\vec{B} = \frac{\mu_0}{4\pi}\int_V \vec{J}\times\frac{\vec{R}}{R^3}\,dV \qquad \text{where } \vec{R} = \vec{r}-\vec{r_0} = \hat{i}(x-x_0)+\hat{j}(y-y_0)+$$

$$\therefore\ \vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int_V \vec{\nabla}\cdot\left[\vec{J}(\vec{r_0})\times\frac{\vec{R}}{R^3}\right] dV(x_0,y_0,z_0).$$

$$= \frac{\mu_0}{4\pi}\int\left[\frac{\vec{R}}{R^3}\cdot(\vec{\nabla}\times\vec{J}) - \vec{J}\cdot\left(\vec{\nabla}\times\frac{\vec{R}}{R^3}\right)\right]dV \qquad \left[\text{as } \vec{J} = \vec{J}(x_0,y_0,\ldots) \quad \boxed{\vec{\nabla}\times\vec{J}=0}\right.$$

$$= -\frac{\mu_0}{4\pi}\int\vec{J}\cdot\left[\frac{1}{R^3}(\vec{\nabla}\times\vec{R}) + \vec{\nabla}\left(\frac{1}{R^3}\right)\times\vec{R}\right]dV \qquad \left[\boxed{\vec{\nabla}\times\vec{R}=0}\right.$$

$$= \frac{\mu_0}{4\pi}\int\frac{3\vec{R}}{R^5}\times\vec{R}\,dV$$

$$= 0$$

Therefore $\boxed{\vec{\nabla}\cdot\vec{B}=0}$ i.e magnetic field is of solenoidal natu

We can prove $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$

Magnetic vector Potential :

[Calculation of electric field is very much simplified with the introduction of electric potential.

Similarly, in magnetostatic calculation mag. field $\vec{B}$ may be simplified with the introduction of magnetic potential.]

* For a magnetic field, we have $\vec{\nabla}\cdot\vec{B} = 0$ ——— ①

Again divergence of curl of any vector is zero i.e $\vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0$ —— ②

Therefore we may write $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ ——— ③

Since $\vec{A}$ is related to magnetic field $\vec{B}$; therefore $\vec{A}$ is called magnetic vector potential

We have Ampere's law $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$

or, $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{J}$

or. $\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$ ——— ④

$\vec{A}$ is so choosen whose divergence is zero with no effect on $\vec{B}$

i.e $\vec{\nabla}\cdot\vec{A} = 0$

Therefore equation ④ can be written as,

$$\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}} \quad \text{——— ⑤}$$

called vector Poisson's equation.

In component form equation ⑤ can be written as,

$$\left.\begin{aligned} \nabla^2 A_x &= -\mu_0 J_x \\ \nabla^2 A_y &= -\mu_0 J_y \\ \nabla^2 A_z &= -\mu_0 J_z \end{aligned}\right\} \text{In general } \nabla^2 A_i = -\mu_0 J_i \text{ ——}$$

where $i = x, y, z$

Solution : Poisson's equation in electrostatics,

$$\nabla^2\phi = -\rho/\epsilon_0$$

which has a solution for volume charge distri[bution]

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}_0)\,dv_0}{r_0}$$

Using this as a guide, the solution of equation ⑥ can be written as,

$$A_i(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{J_i(R_0)\,dV_0}{R} \qquad \text{—— ⑦}$$

where $R = |\vec{r} - \vec{r_0}|$

$$\begin{cases} A_x(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{J_x(R_0)\,dV_0}{R} \\ A_y(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{J_y(R_0)\,dV_0}{R} \\ A_z(\vec{r}) = \dfrac{\mu_0}{4\pi}\displaystyle\int \dfrac{J_z(R_0)\,dV_0}{R} \end{cases}$$

Vectorically

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}(\vec{R_0})\,dV_0}{R}} \qquad \text{—— ⑧}$$

which is magnetic vector potential.

<u>For line current</u> $\vec{J}\,dV_0$ has to be replaced by $i\vec{dl}$

then

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_C \frac{i\,\vec{dl}(\vec{R_0})}{R}} \qquad \text{—— ⑨}$$

<u>Note</u>: Magnetic vector potential $\vec{A}$ is not as useful as electrostatic potential ϕ in calculating magnetic field. Use of $\vec{A}$ in the calculation of $\vec{B}$ is limited.

Its main use is in problems involving electromagnetic radiations.

Derivation of Bio-Savart law from Magnetic vector potential :-

Magnetic vector potential $\vec{A}(\vec{r})$ at a point $\vec{r}(x,y,z)$ due to a volume distribution of current $\vec{J}(\vec{R_0})$ is given by,

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\,dV_0}{R} \qquad \text{—— ①} \quad \text{Where } R = |\vec{r} - \vec{r_0}|$$

Taking curl, we get -

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\frac{1}{R}\vec{J}\,dV_0$$

$$= \frac{\mu_0}{4\pi}\int \left[\vec{\nabla}\left(\frac{1}{R}\right)\times\vec{J} + \frac{1}{R}(\vec{\nabla}\times\vec{J})\right] dV_0$$

$$\therefore \vec{B} = \frac{\mu_0}{4\pi}\int\left(-\frac{\vec{R}}{R^3}\times\vec{J} + 0\right) dv_0 \quad \left\{ \begin{array}{l} \text{as } \vec{J} = \vec{J}(x_0, y_0, z_0) \\ \vec{\nabla}\times\vec{J} = 0. \end{array}\right.$$

or,
$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\times\vec{R}}{R^3}\, dv_0} \quad \text{--- (2)}$$

- In case of a line current $\vec{J}\, dv_0$ is replaced by $i\vec{dl}$.

Thus
$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int_C \frac{i\vec{dl}\times\vec{R}}{R^3}} \quad \text{--- (3)}$$

Equation (2) and (3) are known as Bio-savart law.

Problem: The magnetic induction $\vec{B}$ and magnetic vector potential $\vec{A}$ due to a current element $i\vec{dl}$ are given by, $\vec{B} = \frac{\mu_0}{4\pi}\cdot\frac{i\vec{dl}\times\vec{R}}{R^3}$ and $\vec{A} = \frac{\mu_0}{4\pi}\frac{i\vec{dl}}{R}$ respectively. Calculate $\vec{\nabla}\times\vec{A}$ and $\vec{\nabla}\cdot\vec{B}$.

(i)
$$\vec{\nabla}\times\vec{A} = \vec{\nabla}\times\frac{\mu_0}{4\pi}\frac{i\vec{dl}}{R} \qquad \left[\vec{\nabla}\times\phi\vec{A} = \vec{\nabla}\phi\times\vec{A} + \phi\,\vec{\nabla}\times\vec{A}\right]$$

$$= \frac{\mu_0 i}{4\pi}\left[\vec{\nabla}\left(\frac{1}{R}\right)\times\vec{dl} + \frac{1}{R}(\vec{\nabla}\times\vec{dl})\right] \qquad \left[\vec{\nabla}\times\vec{dl} = 0.\right.$$

$$= \frac{\mu_0 i}{4\pi}\left[-\frac{\vec{R}}{R^3}\times\vec{dl}\right]$$

$$= \frac{\mu_0}{4\pi}\frac{i\vec{dl}\times\vec{R}}{R^3}$$

$$= \vec{B}.$$

(ii)
$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot\left(\frac{\mu_0}{4\pi}\frac{i\vec{dl}\times\vec{R}}{R^3}\right)$$

$$= \frac{\mu_0 i}{4\pi}\vec{\nabla}\cdot\left(\vec{dl}\times\frac{\vec{R}}{R^3}\right) \qquad \left[\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\vec{\nabla}\times\vec{A} - \vec{A}\cdot\vec{\nabla}\times\vec{B}.\right.$$

$$= \frac{\mu_0 i}{4\pi}\left[\frac{\vec{R}}{R^3}\cdot\cancelto{0}{(\vec{\nabla}\times\vec{dl})} - \vec{dl}\cdot\left(\vec{\nabla}\times\frac{\vec{R}}{R^3}\right)\right]$$

$$= -\frac{\mu_0 i}{4\pi}\vec{dl}\cdot\left[\vec{\nabla}\left(\frac{1}{R^3}\right)\times\vec{R} + \frac{1}{R^3}\cancelto{0}{(\vec{\nabla}\times\vec{R})}\right]$$

$$= -\frac{\mu_0 i}{4\pi}\vec{dl}\cdot\left(-\frac{3\vec{R}}{R^5}\times\vec{R}\right)$$

$$= 0$$