Semester-IV

B.Sc (Honours) in Physics

C8T: Mathematical Physics III

Lecture on Matrices

By

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Lecture-II

4. Matrices in graph theory and electrical networks

In modern mathematical language, a *network* is called a graph which consists of a set of n points called *vertices* (or nodes) $a_1, a_2, \ldots, a_n$ and lines (or curves) joining the vertices called *edges* (or *links*). For example, in electrical networks, edges represent wires and vertices are *nodes* where wires are connected. In the framework of a building, edges represent beams and vertices are joints where beams are connected. Graphs or networks have a wide variety of applications from transportation to telecommunications. They have been used effectively as powerful tools in electrical, industrial and civil engineering, communication networks in the planning of business and industry.

Matrices play a significant role in combinatorial problems of many different kinds and, in particular, in graph theory and in electrical networks. They serve as a very useful tool in the study of graphs or networks. This short section deals with some examples of applications of matrices to study graphs of different kinds.[4,5]

In general, the matrix notation a_{ij} is often used in graph theory to denote edges from the ith vertex to the jth vertex, and a_{ji} to denote the number of edges from the jth vertex to the ith vertex. Sometimes, the edges in a graph represent connections that only operate in one direction so that arrows are used to indicate these directions. A graph of this type is called a *directed graph* (or simply, *digraph*). It is very convenient to store a large graph in a computer in the matrix form and then use a modern computer program, e.g. MATLAB, to perform matrix computations.

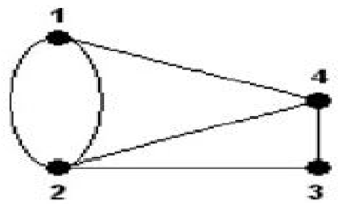
An adjacency matrix $A = (a_{ij})$ of a graph G is defined by

$$a_{ij} = \begin{cases} r, & \text{if vertices } i \text{ and } j \text{ are joined by } r \text{ edges} \\ 0, & \text{otherwise} \end{cases}.$$

A graph G of four vertices and six edges is shown in Figure 1(a) and the adjacency matrix A of the graph G is given by Figure 1(b). Another adjacency matrix A of a star graph G of five vertices and five edges is given in Figure 2(a) and 2(b). An *adjacency matrix* $A = (a_{ij})$ of a digraph G in which there is exactly one directed edge between every pair of vertices is defined by

$$a_{ij} = \begin{cases} 1, & \text{if vertices } i \text{ and } j \text{ are joined by an arrow from } i \text{ to } j \\ 0, & \text{otherwise} \end{cases}.$$

The adjacency matrix A of the digraph G in Figure 3(a) is given in Figure 3(b).

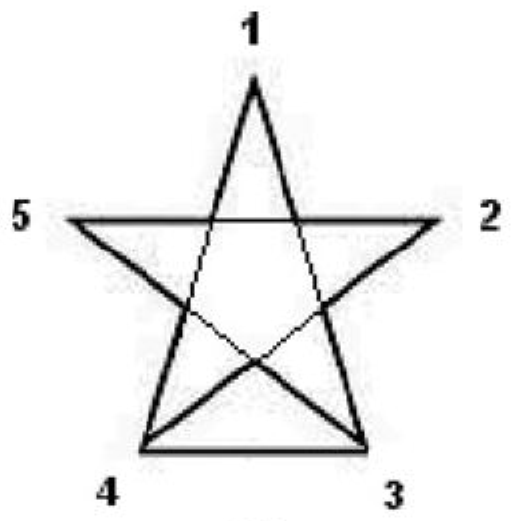


(a)

$$A = \begin{matrix} & 1 & 2 & 3 & 4 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

(b)

Figure 1. (a) Graph G of four vertices and six edges and (b) its 4×4 adjacency matrix A.

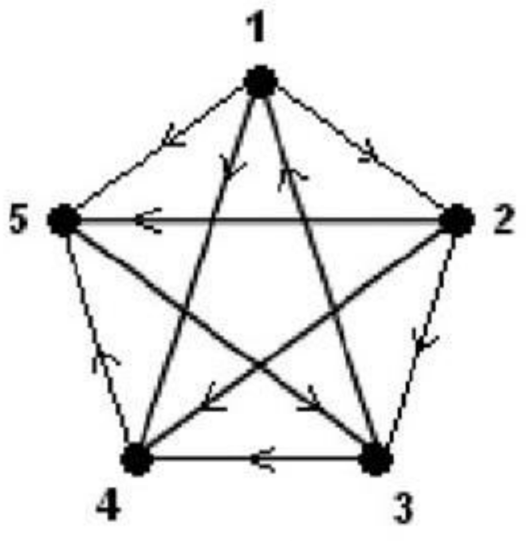


(a)

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

(b)

Figure 2. (a) Star graph G of five vertices and five edges and (b) its 5×5 adjacency matrix A.



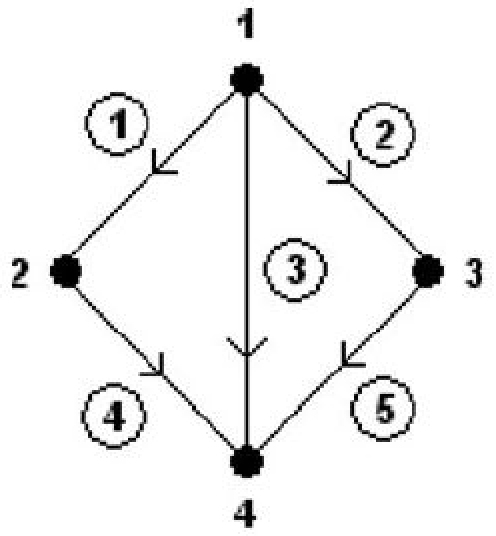
(a)

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

(b)

Figure 3. (a) Digraph of a tournament of five players 1–5 and (b) its 5×5 adjacency matrix A.

The incidence matrix $A = (a_{mn})$ associated with a digraph G consisting of n vertices connected by m edges is an $m \times n$ matrix whose rows are indexed by the edges and whose columns are indexed by the vertices. If edge r starts at vertex i and ends at vertex j, then the rth row of the incidence matrix will have ± 1 in its (r, j) entry and -1 in its (r, j) entry; all



(a)

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

(b)

Figure 4. (a) A simple digraph and (b) its 5×4 incidence matrix A.

other entries in the row are zero. Thus, the element ± 1 corresponds to the outgoing vertex at which the edge starts, and the element -1 to the incoming vertex at which it ends. The incidence matrix A of a digraph G in Figure 4(a), consisting of five edges joined at four different vertices, is of size 5×4 and is given in Figure 4(b).

5. Applications of Matrices

Matrices have many applications in diverse fields of science, commerce and social science. Matrices are used in

(i) Computer Graphics
(ii) Optics
(iii) Cryptography
(iv) Economics
(v) Chemistry
(vi) Geology
(vii) Robotics and animation
(viii) Wireless communication and signal processing
(ix) Finance ices

5.1 Use of Matrices in Computer Graphics

Earlier architecture, cartoon, automation were done by hand drawings but nowadays they are done by using computer graphics. In video gaming industry matrices are major mathematical tool to construct and manipulate a realistic animation of a polygonal figure. Computer graphics software uses matrices to process linear transformations to translate images. For this purpose square matrices are very easily represent linear transformation of objects. Matrices are used to project three dimensional images into two dimensional planes. In Graphics, digital image is treated as a matrix to be start with. The rows and columns of matrix correspond to rows and columns of pixels and the numerical entries correspond to the pixels' color values. Using matrices to manipulate a point is common mathematical approach in video game graphics

Matrices are used to express graphs. Every graph can be representing as a matrix each column and each row of a matrix is node and value of their intersection is strength of the connection between them. Matrix operations such as translation, rotation and sealing are used in graphics. For transformation of a point we use the equation

TRANSFORMED POINT = TRANSFORMATION MATRIX * ORIGINAL POINT

5.2 Use of matrices in cryptography

Cryptography is the technique to encrypting data so that only the relevant person can get the data and relate information. In earlier days, video signals were not used to encrypt. Anyone with satellite dish was able to watch videos which results in the loss for satellite owners, so they started encrypting the video signals so that only those who have videos cipher can unencryptedthe signals.

This encrypting is done by using an invertible key is not invertible then the encrypted signals cannot be unencrypted and they cannot get back to original form. This process is done using matrices. A digital audio or video signal is firstly taken as a sequence of numbers representing the variation over time of air pressure of an acoustic audio signal. The filtering techniques are used which depends on matrix multiplication.

Consider the message "Do Not Worry" .The message is converted into a sequence of numbers from 1 to 26. For space use digit 0.

i.e.

Let

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

The message "DO NOT WORRY" can be encoded as sequence of numbers

4 15 0 14 15 20 0 23 15 18 18 25

This data is placed into matrix

$$A = \begin{bmatrix} 4 & 15 \\ 0 & 14 \\ 15 & 20 \\ 0 & 23 \\ 15 & 18 \\ 18 & 25 \end{bmatrix}$$

To encrypt this data invertible matrix is used; choose a matrix whose determinant in non-zero and whose multiplication is possible with matrix A. The choice of this matrix depends on the person who is encrypting data.

Suppose we use an invertible matrix

$$B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

$$\text{Then } X = AB = \begin{bmatrix} 4 & 15 \\ 0 & 14 \\ 15 & 20 \\ 0 & 23 \\ 15 & 18 \\ 18 & 25 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 53 & 64 \\ 42 & 56 \\ 90 & 95 \\ 69 & 92 \\ 84 & 89 \\ 111 & 118 \end{bmatrix}$$

Now the message that will pass in air to the other person is 53 64 42 56 90 95 69 92 84 89 111 118. To read the original message one needs the key that is B and its inverse. There for to unencrypt data first we will find B^{-1}

$$B^{-1} = \begin{bmatrix} 0.8 & -0.2 \\ -0.6 & 0.4 \end{bmatrix}$$

The original message can be read by only that person who has this invertible key B.

To get original message we operate B^{-1} on AB =X

$(AB)\, B^{-1} = X\, B^{-1}$

$A\, (BB^{-1}) = X\, B^{-1}$

$AI = X\, B^{-1}$

$$A = \begin{bmatrix} 53 & 64 \\ 42 & 56 \\ 90 & 95 \\ 69 & 92 \\ 84 & 89 \\ 111 & 118 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & -0.2 \\ -0.6 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 15 \\ 0 & 14 \\ 15 & 20 \\ 0 & 23 \\ 15 & 18 \\ 18 & 25 \end{bmatrix}$$

There for matrix A is obtained back and message can be rewritten as 4 15 0 14 15 20 0 23 15 18 18 23.

5.3 Use of Matrices in Wireless Communication

Matrices are used to model the wireless signals and to optimize them. For detection, extractions and processing of the information embedded in signals matrices one used. Matrices play a key role in signal estimation and detection problems. They are used in sensor array signal processing and design of adaptive filter. Matrices play a major role in representing and processing digital images. We know that wireless and communication is important part of telecommunication industry. Sensor array signal processing focuses on signal enumeration and source location applications and presents a huge importance in many domains such as radar signals and underwater surveillance. Main problem in sensor array signal processing is to detect and locate the radiating sources given the temporal and spatial information collected from the sensors.

5.4 Use of Matrices in Economics

Matrix Cramer's Rule and determinants are simple and important tools for solving many problems in business and economics related to maximize profit and minimize loss. Matrices are used to find variance and covariance. Matrix Cramer's Rule is used to find solutions of linear equations with the help of matrix determinant. The equilibrium of markets in IS-LM model is solved by using determinants and Matrix Cramer's Rule.

5.5 Matrices for finding area of triangle

Matrices can be used to find area of any triangle whose vertices are given. Suppose vertices of the triangle Δ ABC are A (a, b), B(c, d) C (e, f). Then area of Δ ABC is given by the following determinant

Area of Δ ABC $= \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$

5.6 Matrices for collinear points

Matrices are use to test whether the given three points are collinear. If A (a, b), B(c, d) C (e, f) are three given points in plane. Then these points are collinear if they are unable to form a triangle. I.e. area of triangle formed by A, B, C should be zero

∴ A, B, C are collinear if $\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$ vanishes.

5.7 Matrices for Solution of Linear Equations

Matrices are used to solve system of linear equations. Cramer's rule is used for this purpose.

What is Cramer's Rule?

We can express system of linear equations in formof matrices

If we have linear equation

ax+by=c

dx+ey=fthen we can express these equations in matrix form as AX=B

here A is coefficient matrix with value $A = \begin{bmatrix} a & b \\ d & e \end{bmatrix}$, X is variable matrix with value $X = \begin{bmatrix} x \\ y \end{bmatrix}$

and B is right hand side of linear system with value $B = \begin{bmatrix} c \\ f \end{bmatrix}$, then by Cramer's rule

$x = \frac{|C|}{A}$ $y = \frac{|D|}{A}$ here $C = \begin{bmatrix} c & b \\ f & e \end{bmatrix}$ and $D = \begin{bmatrix} a & c \\ d & f \end{bmatrix}$

5.8 Matrices for Financial Records

Matrices allow to represent array of many numbers as a single object and is denoted by a single symbol then calculations are performed on these symbols in very compact form. The matrix method of obtaining opening and closing balances for any accounting period is very efficient, accurate and less time consuming.

5.9 Matrices for Engineering

Matrices applications involve the use of eigen values and eigen vectors in the process of transforming a given matrix into a diagonal matrix. Linear algebra is useful tool for solving large number of variables in such a short time. It is interesting to note that many of the calculus theorems used in engineering classes are proved quickly and easily through linear algebra. Transformation matrices are commonly used in computer graphics and image processing. Matrices are used in computer generated images that has a reflection and distortion effect such as high passing through ripping water. Used to calculate the electrical properties of a circuit with voltage and enrage, resistance and to calculate battery power output.

Matrices are used in realistic looking motion on a two dimensional computer screen and calculations in algorithms that create Google page ranking. They are also used for compressing electronic information and storing fingerprints information. Errors in electronic transmissions are identified and corrected with the use of matrices.

Movements of the robots are programmed with the calculations of matrices rows and columns. The inputs for controlling robots are based on calculations from matrices.

5.10 Matrices for Physics

Matrices are used in science of optics to account for reflection and for refraction. Matrices are also useful in electrical circuits and quantum mechanics and resistar conversion of electrical energy. Matrices are used to solve AC network equations in electric circuits.

References:

[1] L. Debnath, A brief historical introduction to matrices and their applications, International Journal of Mathematical Education in Science and Technology (2014) 45 (3), pp. 360-377, DOI: 10.1080/0020739X.2013.837521

[2] S Kaur, Applications of Matrices, International Journal of Engineering Technology Science and Research (2017) 4 (11), pp. 284-288, ISSN 2394 – 3386.

Selected Questions on Matrix

1. Apply Laplace's method to show that

$$\begin{vmatrix} 0 & -c & b & x \\ c & 0 & -a & y \\ -b & a & 0 & z \\ x' & y' & z' & w \end{vmatrix} = -(ax+by+cz)(ax'+by'+cz').$$

2. Show that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix} = -(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta).$$

3. Solve:

$$\begin{aligned} x - 2y + 3z &= 5 \\ 2x - y - z &= 1 \\ x + y + z &= 3. \end{aligned}$$

4. If $A = (a_{ij})_{3\times 3}$ be an orthogonal matrix and

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= 1 \\ a_{21}x + a_{22}y + a_{23}z &= 0 \\ a_{31}x + a_{32}y + a_{33}z &= 0 \end{aligned}$$

then show that $= a_{11}$, $y = a_{12}$, $z = a_{13}$.

5. Reduced the following matrix to a row echelon matrix by applying elementary row operation

$$A = \begin{pmatrix} 3 & 0 & 6 & 9 \\ 2 & 5 & -6 & -4 \\ 7 & 2 & 8 & 5 \\ 0 & 3 & -9 & -24 \end{pmatrix}.$$

6. Reduced the matrix

$$A=\begin{bmatrix}1 & 2 & 1 & 2\\ 1 & 3 & 2 & 2\\ 2 & 4 & 3 & 4\\ 3 & 7 & 4 & 6\end{bmatrix}$$

to the normal form and hence, determine its rank.

7. Find two non-singular matrices P and Q such that PAQ is in the normal form where

$$A=\begin{pmatrix}4 & 1 & -2 & 1\\ 3 & -1 & 1 & 2\\ 1 & 2 & -1 & 3\\ 1 & 2 & 0 & 1\end{pmatrix}.$$

Also determine the rank A.

8. Find the inverse of the matrix

$$\begin{pmatrix}0 & 1 & 2 & 2\\ 1 & 1 & 2 & 3\\ 2 & 2 & 2 & 3\\ 2 & 3 & 3 & 3\end{pmatrix}$$

by using elementary operations.

9. Show that the only real value of λ for which the following equations have non-zero solution is 6.

$$x+2y+3z=\lambda x$$
$$3x+y+2z=\lambda y$$
$$2x+3y+z=\lambda z.$$

10. Discuss the nature of the solutions, for all values of k, of the following equations,

$$2x+3ky+(3k+4)z=0$$
$$x+(k+4)y+(4k+2)z=0$$
$$x+2(k+1)y+(3k+4)z=0.$$

11. Show that the equation

$$-2x+y+z=a$$
$$x-2y+z=b$$
$$x+y-2z=c$$

have no solution unless $a+b+c=0$, in which case they have infinitely many solutions. Find these solution when $a=1$, $b=1$, $c=-2$.

12.Determine the condition for which the system

$$x + y + z = 1$$
$$x + 2y - z = \mu$$
$$5x + 7y + \alpha z = \mu^2$$

admits of (i) infinite number of solutions, (ii) no solution, (iii) a unique solution.

13.For what values of k the system of equations is consistent and solve in each consistent case

$$x + y + z = 1$$
$$2x + 3y - z = k + 1$$
$$2x + y + 5z = k^2 + 1.$$

14. Given that $AB = AC$. Does it follow that $= C$? If so, then state the condition for A. Given a counter Example.

15. If $AB = 0$ where A, B non-zero matrices, then show that both A and B are singular matrices.

16. If A is a non-singular matrix of order n, then show that $adj(adj\ A) = |A|^{n-2}A$. Verify it, if $= \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$.

17.Solve the system of homogeneous equations

$$2x_1 + x_2 - 3x_3 - x_4 = 0$$
$$x_1 + x_3 + 2x_4 = 0$$
$$5x_1 - x_2 - 2x_3 + x_4 = 0.$$

18. Solve the system of non-homogeneous equations

$$2x_1 + x_3 + x_4 = 1$$
$$x_2 + 2x_4 = -2$$
$$x_1 + x_2 + 2x_3 = -3.$$