Semester-IV

B.Sc (Honours) in Physics

C8T: Mathematical Physics III

**Lecture**

**on**

# Matrices

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**Lecture-III**

**(Specially: Eigen-values and Eigenvectors )**

# Syllabus

- ❑ **Eigen-values and Eigenvectors**
- ❑**Cayley- Hamiliton Theorem.**
- ❑ **Diagonalization of Matrices.**
- ❑ **Solutions of Coupled Linear Ordinary Differential Equations.**
- ❑ **Functions of a Matrix.**

# Eigen-values and Eigenvectors.

## Definitions

Consider the matrix $\mathbf{A}$ and the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ given by

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

Forming the products $\mathbf{Ax}_1$, $\mathbf{Ax}_2$, and $\mathbf{Ax}_3$, we obtain

$$\mathbf{Ax}_1 = \begin{bmatrix} 8 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{Ax}_2 = \begin{bmatrix} 9 \\ 6 \\ 6 \end{bmatrix}, \quad \mathbf{Ax}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

But

$$\begin{bmatrix} 8 \\ 2 \\ 0 \end{bmatrix} = 2\mathbf{x}_1, \quad \begin{bmatrix} 9 \\ 6 \\ 6 \end{bmatrix} = 3\mathbf{x}_2, \quad \text{and} \quad \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 1\mathbf{x}_3;$$

hence,

$$\begin{aligned} \mathbf{Ax}_1 &= 2\mathbf{x}_1, \\ \mathbf{Ax}_2 &= 3\mathbf{x}_2, \\ \mathbf{Ax}_3 &= 1\mathbf{x}_3. \end{aligned}$$

That is, multiplying $\mathbf{A}$ by any one of the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, or $\mathbf{x}_3$ is equivalent to simply multiplying the vector by a suitable scalar.

**Definition 1** A nonzero vector $\mathbf{x}$ is an *eigenvector* (or characteristic vector) of a square matrix $\mathbf{A}$ if there exists a scalar $\lambda$ such that $\mathbf{Ax} = \lambda\mathbf{x}$. Then $\lambda$ is an *eigenvalue* (or characteristic value) of $\mathbf{A}$.

Thus, in the above example, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are eigenvectors of $\mathbf{A}$ and 2, 3, 1 are eigenvalues of $\mathbf{A}$.

Note that eigenvectors and eigen values are only defined for square matrices. Furthermore, note that the zero vector can not be an eigenvector even though $A \cdot 0 = \lambda \cdot 0$ for every scalar $\lambda$. An eigen value, however, can be zero.

**Example 1** Show that

$$\mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

is an eigenvector of

$$\mathbf{A} = \begin{bmatrix} 0 & 5 & 7 \\ 0 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}.$$

**Solution**

$$\mathbf{Ax} = \begin{bmatrix} 0 & 5 & 7 \\ 0 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

Thus, $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ and $\lambda = 0$ is an eigenvalue.

**Example 2** Is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

an eigenvector of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}?$$

**Solution**

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Thus, if $\mathbf{x}$ is to be an eigenvector of $\mathbf{A}$, there must exist a scalar $\lambda$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, or such that

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}.$$

It is quickly verified that no such $\lambda$ exists, hence $\mathbf{x}$ is not an eigenvector of $\mathbf{A}$.

# Problems

**1.** Determine which of the following vectors are eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -4 & 7 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, (c) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

(e) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, (f) $\begin{bmatrix} -4 \\ -4 \end{bmatrix}$, (g) $\begin{bmatrix} 4 \\ -4 \end{bmatrix}$, (h) $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

**2.** What are the eigenvalues that correspond to the eigenvectors found in Problem 1?

**3.** Determine which of the following vectors are eigenvectors for

$$\mathbf{B} = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, (c) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

(e) $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$, (f) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, (g) $\begin{bmatrix} -4 \\ -6 \end{bmatrix}$, (h) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

**4.** What are the eigenvalues that correspond to the eigenvectors found in Problem 3?

**5.** Determine which of the following vectors are eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 2 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, (c) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$,

(e) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, (f) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, (g) $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, (h) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

6. What are the eigenvalues that correspond to the eigenvectors found in Problem 5?

7. Determine which of the following vectors are eigenvectors for

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 3 \end{bmatrix}.$$

(a) $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, (c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$,

(d) $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, (e) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, (f) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

8. What are the eigenvalues that correspond to the eigenvectors found in Problem 7?

# Eigen values

Let $\mathbf{x}$ be an eigenvector of the matrix $\mathbf{A}$. Then there must exist an eigenvalue $\lambda$ such that

$$\mathbf{Ax} = \lambda \mathbf{x} \tag{1}$$

or, equivalently,

$$\mathbf{Ax} - \lambda \mathbf{x} = \mathbf{0}$$

or

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}. \tag{2}$$

CAUTION. We could not have written (2) as $(\mathbf{A} - \lambda)\mathbf{x} = \mathbf{0}$ since the term $\mathbf{A} - \lambda$ would require subtracting a scalar from a matrix, an operation which is not defined. The quantity $\mathbf{A} - \lambda\mathbf{I}$, however, is defined since we are now subtracting one matrix from another.

Define a new matrix

$$\mathbf{B} = \mathbf{A} - \lambda \mathbf{I}. \tag{3}$$

Then (2) may be rewritten as

$$\mathbf{Bx} = \mathbf{0}, \tag{4}$$

is a linear homogeneous system of equations for the unknown $\mathbf{x}$. If $\mathbf{B}$ has an inverse, then we can solve Eq. (4) for $\mathbf{x}$, obtaining $\mathbf{x} = \mathbf{B}^{-1}\mathbf{0}$, or $\mathbf{x} = \mathbf{0}$. This result, however, is absurd since $\mathbf{x}$ is an eigenvector and cannot be zero. Thus, it follows that $\mathbf{x}$ will be an eigenvector of $\mathbf{A}$ if and only if $\mathbf{B}$ does not have an inverse. But if a square matrix does not have an inverse, then its determinant must be zero . Therefore, $\mathbf{x}$ will be an eigenvector of $\mathbf{A}$ if and only if

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0. \tag{5}$$

Equation (5) is called the *characteristic equation of* $\mathbf{A}$. The roots of (5) determine the eigenvalues of $\mathbf{A}$.

**Example 1** Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

**Solution**

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}.$$

$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5$. The characteristic equation of $\mathbf{A}$ is $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, or $\lambda^2 - 4\lambda - 5 = 0$. Solving for $\lambda$, we have that $\lambda = -1, 5$; hence the eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1$, $\lambda_2 = 5$.

**Example 2** Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}.$$

**Solution**

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix},$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(1-\lambda) + 2 = \lambda^2 - 2\lambda + 3.$$

The characteristic equation is $\lambda^2 - 2\lambda + 3 = 0$; hence, solving for $\lambda$ by the quadratic formula, we have that $\lambda_1 = 1 + \sqrt{2}\,i$, $\lambda_2 = 1 - \sqrt{2}\,i$ which are eigenvalues of $\mathbf{A}$.

**Note: Even if the elements of a matrix are real, the eigen values may be complex**

**Example 3** Find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} t & 2t \\ 2t & -t \end{bmatrix}.$$

**Solution**

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} t & 2t \\ 2t & -t \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} t-\lambda & 2t \\ 2t & -t-\lambda \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (t-\lambda)(-t-\lambda) - 4t^2 = \lambda^2 - 5t^2.$$

The characteristic equation is $\lambda^2 - 5t^2 = 0$, hence, the eigenvalues are $\lambda_1 = \sqrt{5}t$, $\lambda_2 = -\sqrt{5}t$.

NOTE: If the matrix **A** depends on a parameter (in this case the parameter is $t$), then the eigenvalues may also depend on the parameter.

**Example 4** Find the eigenvalues for

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Solution**

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & -1 & 1 \\ 3 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}.$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)[(2-\lambda)(-2-\lambda) + 3] = (1-\lambda)(\lambda^2 - 1).$$

The characteristic equation is $(1-\lambda)(\lambda^2 - 1) = 0$; hence, the eigenvalues are $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$.

NOTE: The roots of the characteristic equation can be repeated. That is, $\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_k$. When this happens, the eigenvalue is said to be of *multiplicity* $k$. Thus, in Example 4, $\lambda = 1$ is an eigenvalue of multiplicity 2 while, $\lambda = -1$ is an eigenvalue of multiplicity 1.

From the definition of the characteristic Equation (5), it can be shown that if $\mathbf{A}$ is an $n \times n$ matrix then the characteristic equation of $\mathbf{A}$ is an $n$th degree polynomial in $\lambda$. It follows from the Fundamental Theorem of Algebra, that the characteristic equation has $n$ roots, counting multiplicity. Hence, $\mathbf{A}$ has exactly $n$ eigenvalues, counting multiplicity. (See Examples 1 and 4).

In general, it is very difficult to find the eigenvalues of a matrix. First the characteristic equation must be obtained, and for matrices of high order this is a lengthy task. Then the characteristic equation must be solved for its roots. If the equation is of high order, this can be an impossibility in practice. For example, the reader is invited to find the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}.$$

For these reasons, eigenvalues are rarely found by the method just given, and numerical techniques are used to obtain approximate values

## Problems

In Problems 1 through 35, find the eigen values of the given matrices

**1.** $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, **2.** $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, **3.** $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$,

**4.** $\begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix}$, **5.** $\begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$, **6.** $\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$,

**7.** $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$, **8.** $\begin{bmatrix} 3 & 5 \\ -5 & -3 \end{bmatrix}$, **9.** $\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$,

**10.** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, **11.** $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, **12.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

**13.** $\begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix}$, **14.** $\begin{bmatrix} 4 & 10 \\ 9 & -5 \end{bmatrix}$, **15.** $\begin{bmatrix} 5 & 10 \\ 9 & -4 \end{bmatrix}$,

**16.** $\begin{bmatrix} 0 & t \\ 2t & -t \end{bmatrix}$, **17.** $\begin{bmatrix} 0 & 2t \\ -2t & 4t \end{bmatrix}$, **18.** $\begin{bmatrix} 4\theta & 2\theta \\ -\theta & \theta \end{bmatrix}$,

**19.** $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$, **20.** $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 2 \\ -1 & 0 & 2 \end{bmatrix}$, **21.** $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

**22.** $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$, **23.** $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 6 & 4 \\ 2 & 3 & 5 \end{bmatrix}$, **24.** $\begin{bmatrix} 5 & -7 & 7 \\ 4 & -3 & 4 \\ 4 & -1 & 2 \end{bmatrix}$,

**25.** $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$, **26.** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$, **27.** $\begin{bmatrix} 10 & 2 & 0 \\ 2 & 4 & 6 \\ 0 & 6 & 10 \end{bmatrix}$,

**28.** $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, **29.** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, **30.** $\begin{bmatrix} 4 & 2 & 1 \\ 2 & 7 & 2 \\ 1 & 2 & 4 \end{bmatrix}$,

**31.** $\begin{bmatrix} 1 & 5 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, **32.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, **33.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 27 & -27 & 9 \end{bmatrix}$,

**34.** $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, **35.** $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{bmatrix}$.

## Eigenvectors

To each distinct eigenvalue of a matrix **A** there will correspond at least one eigenvector which can be found by solving the appropriate set of homogeneous equations. If an eigenvalue $\lambda_i$ is substituted into (2), the corresponding eigenvector $\mathbf{x}_i$ is the solution of

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x}_i = \mathbf{0}. \tag{6}$$

**Example 1** Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

**Solution** The eigenvalues of **A** have already been found to be $\lambda_1 = -1,\ \lambda_2 = 5$ (see Example 1 of Section 6.2). We first calculate the eigenvectors corresponding to $\lambda_1$. From (6),

$$(\mathbf{A} - (-1)\mathbf{I})\mathbf{x}_1 = \mathbf{0}. \tag{7}$$

If we designate the unknown vector $\mathbf{x}_1$ by

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix},$$

Eq. (7) becomes

$$\left\{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or, equivalently,

$$2x_1 + 2y_1 = 0,$$
$$4x_1 + 4y_1 = 0.$$

A nontrivial solution to this set of equations is $x_1 = -y_1$, $y_1$ arbitrary; hence, the eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -y_1 \\ y_1 \end{bmatrix} = y_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad y_1 \text{ arbitrary.}$$

By choosing different values of $y_1$, different eigenvectors for $\lambda_1 = -1$ can be obtained. Note, however, that any two such eigenvectors would be scalar multiples of each other, hence linearly dependent. Thus, there is only one linearly independent eigenvector corresponding to $\lambda_1 = -1$. For convenience we choose $y_1 = 1$, which gives us the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Many times, however, the scalar $y_1$ is chosen in such a manner that the resulting eigenvector becomes a unit vector. If we wished to achieve this result for the above vector, we would have to choose $y_1 = 1/\sqrt{2}$.

Having found an eigenvector corresponding to $\lambda_1 = -1$, we proceed to find an eigenvector $\mathbf{x}_2$ corresponding to $\lambda_2 = 5$. Designating the unknown vector $\mathbf{x}_2$ by

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

and substituting it with $\lambda_2$ into (6), we obtain

$$\left\{\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix} - 5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\}\begin{bmatrix}x_2\\ y_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix},$$

or

$$\begin{bmatrix}-4 & 2\\ 4 & -2\end{bmatrix}\begin{bmatrix}x_2\\ y_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix},$$

or, equivalently,

$$\begin{aligned}-4x_2 + 2y_2 &= 0,\\ 4x_2 - 2y_2 &= 0.\end{aligned}$$

A nontrivial solution to this set of equations is $x_2 = \frac{1}{2}y_2$, where $y_2$ is arbitrary; hence

$$\mathbf{x}_2 = \begin{bmatrix}x_2\\ y_2\end{bmatrix} = \begin{bmatrix}y_2/2\\ y_2\end{bmatrix} = y_2\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}.$$

For convenience, we choose $y_2 = 2$, thus

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

In order to check whether or not $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda_2 = 5$, we need only check if $\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$:

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda_2\mathbf{x}_2.$$

Again note that $\mathbf{x}_2$ is *not* unique! Any scalar multiple of $\mathbf{x}_2$ is also an eigenvector corresponding to $\lambda_2$. However, in this case, there is just one *linearly independent* eigenvector corresponding to $\lambda_2$.

**Example 2** Find the eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix}.$$

**Solution** By using the method of the previous section, we find the eigenvalues to be $\lambda_1 = 2$, $\lambda_2 = i$, $\lambda_3 = -i$. We first calculate the eigenvectors corresponding to $\lambda_1 = 2$. Designate $\mathbf{x}_1$ by

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}.$$

Then (6) becomes

$$\left\{ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or, equivalently,

$$\begin{aligned} 0 &= 0, \\ 5z_1 &= 0, \\ -y_1 - 4z_1 &= 0. \end{aligned}$$

A nontrivial solution to this set of equations is $y_1 = z_1 = 0,\ x_1$ arbitrary; hence

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We now find the eigenvectors corresponding to $\lambda_2 = i$. If we designate $\mathbf{x}_2$ by

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix},$$

Eq. (6) becomes

$$\begin{bmatrix} 2-i & 0 & 0 \\ 0 & 2-i & 5 \\ 0 & -1 & -2-i \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} (2-i)x_2 &= 0, \\ (2-i)y_2 + 5z_2 &= 0, \\ -y_2 + (-2-i)z_2 &= 0. \end{aligned}$$

A nontrivial solution to this set of equations is $x_2 = 0$, $y_2 = (-2-i)z_2$, $z_2$ arbitrary; hence,

$$\mathbf{x}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ (-2-i)z_2 \\ z_2 \end{bmatrix} = z_2 \begin{bmatrix} 0 \\ -2-i \\ 1 \end{bmatrix}.$$

The eigenvectors corresponding to $\lambda_3 = -i$ are found in a similar manner to be

$$\mathbf{x}_3 = z_3 \begin{bmatrix} 0 \\ -2-i \\ 1 \end{bmatrix}, \quad z_3 \text{ arbitrary}.$$

It should be noted that even if a mistake is made in finding the eigenvalues of a matrix, the error will become apparent when the eigenvectors corresponding to the incorrect eigenvalue are found. For instance, suppose that $\lambda_1$ in Example 2 was calculated erroneously to be 3. If we now try to find $\mathbf{x}_1$ we obtain the equations.

$$\begin{aligned} -x_1 &= 0, \\ -y_1 + 5z_1 &= 0, \\ -y_1 - 5z_1 &= 0. \end{aligned}$$

The only solution to this set of equations is $x_1 = y_1 = z_1 = 0$, hence

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

However, by definition, an eigenvector cannot be the zero vector. Since every eigenvalue must have a corresponding eigenvector, there is a mistake. A quick check shows that all the calculations above are valid, hence the error must lie in the value of the eigenvalue.

**Problems**

In Problems 1 through 23, find an eigenvector corresponding to each eigen value of the given matrix.

1. $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$,

2. $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$,

3. $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$,

4. $\begin{bmatrix} 3 & 6 \\ 9 & 6 \end{bmatrix}$,

5. $\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$,

6. $\begin{bmatrix} 3 & 5 \\ 5 & -3 \end{bmatrix}$,

7. $\begin{bmatrix} 3 & 5 \\ -5 & -3 \end{bmatrix}$,

8. $\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$,

9. $\begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix}$,

10. $\begin{bmatrix} 4 & 10 \\ 9 & -5 \end{bmatrix}$,

11. $\begin{bmatrix} 0 & t \\ 2t & -t \end{bmatrix}$,

12. $\begin{bmatrix} 4\theta & 2\theta \\ -\theta & \theta \end{bmatrix}$,

13. $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$,

14. $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 2 \\ -1 & 0 & 2 \end{bmatrix}$,

15. $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 2 \\ -1 & 0 & 3 \end{bmatrix}$,

16. $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 6 & 4 \\ 2 & 3 & 5 \end{bmatrix}$,

17. $\begin{bmatrix} 5 & -7 & 7 \\ 4 & -3 & 4 \\ 4 & -1 & 2 \end{bmatrix}$,

18. $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$,

**19.** $\begin{bmatrix} 1 & 5 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, **20.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, **21.** $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 0 \\ 0 & 1 & 5 \end{bmatrix}$,

**22.** $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, **23.** $\begin{bmatrix} 2 & 4 & 2 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 3 & -1 \\ 0 & 2 & 0 & 4 \end{bmatrix}$.

**24.** Find unit eigenvectors (i.e., eigenvectors whose magnitudes equal unity) for the matrix in Problem 1.

**25.** Find unit eigenvectors for the matrix in Problem 2.

**26.** Find unit eigenvectors for the matrix in Problem 3.

**27.** Find unit eigenvectors for the matrix in Problem 13.

**28.** Find unit eigenvectors for the matrix in Problem 14.

**29.** Find unit eigenvectors for the matrix in Problem 16.

**30.** A nonzero vector $\mathbf{x}$ is a left eigenvector for a matrix $\mathbf{A}$ if there exists a scalar $\lambda$ such that $\mathbf{xA} = \lambda\mathbf{x}$. Find a set of left eigenvectors for the matrix in Problem 1.