- SEMESTER –VI (HONOURS)
- PAPER : CC13T (EM THEORY)
- TOPIC: MAXWELLS EQUATIONS AND ITS APPLICATION

CLASS NOTES :BY TAPAS KUMAR CHANDA

Dept. of physics, Bhatter College,Dantan

March 29, 2020

Sub topics: displacement current,em wave in dicelectric medium, poynting theorem , poynting vector.

Next class: EM Energy density ,Momentum density

The displacement current

The calculation of the magnetic field of a current distribution can, in principle, be carried out using Ampere's law which relates the path integral of the magnetic field around a closed path to the current intercepted by an arbitrary surface that spans this path:

$$\int_{path} \bar{B} \cdot d\bar{L} = \mu_0 I$$

Ampere's law is independent of the shape of the surface chosen as long as the current flows along a continuous, unbroken circuit. However, consider the case in which the current wire is broken and connected to a parallel-plate capacitor (see Figure 35.1). A current will flow through the wire during the charging process of the capacitor. This current will generate a magnetic field and if we are far away from the capacitor, this field should be very similar to the magnetic field produced by an infinitely long, continuous, wire. However, the current intercepted by an arbitrary surface now depends on the surface chosen. For example, the surface shown in Figure 35.1 does not intercept any current. Clearly, Ampere's law can not be applied in this case to find the magnetic field generated by the current.

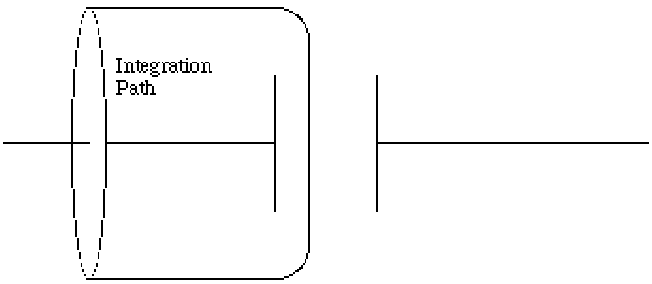

Figure 35.1. Ampere's law in a capacitor circuit.

Although the surface shown in Figure 35.1 does not intercept any current, it intercepts electric flux. Suppose the capacitor is an ideal capacitor, with a homogeneous electric field E between the plates and no electric field outside the plates. At a certain time t the charge on the capacitor plates is Q. If the plates have a surface area A then the electric field between the plates is equal to

$$E = \frac{Q}{\varepsilon_0 A}$$

The electric field outside the capacitor is equal to zero. The electric flux, [Phi]$_E$, intercepted by the surface shown in Figure 35.1 is equal to

$$\Phi_E = E A = \frac{Q}{\varepsilon_0}$$

If a current I is flowing through the wire, then the charge on the capacitor plates will be time dependent. The electric flux will therefore also be time dependent, and the rate of change of electric flux is equal to

$$\frac{d\Phi_E}{dt} = \frac{1}{\varepsilon_0}\frac{dQ}{dt} = \frac{I}{\varepsilon_0}$$

The magnetic field around the wire can now be found by modifying Ampere's law

$$\int_{Path} \bar{B} \cdot d\bar{L} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

where [Phi]$_E$ is the electric flux through the surface indicated in Figure 35.1 In the most general case, the surface spanned by the integration path of the magnetic field can intercept current and electric flux. In such a case, the effects of the electric flux and the electric current must be combined, and Ampere's law becomes

$$\int_{Path} \bar{B} \cdot d\bar{L} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

The current I is the current intercepted by whatever surface is used in the calculation, and is not necessarily the same as the current in the wires. Equation (35.6) is frequently written as

$$\int_{Path} \bar{B} \cdot d\bar{L} = \mu_0 (I + I_d)$$

where I_d is called **the displacement current** and is defined as

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$$

Propagation in a dielectric medium

Consider the propagation of an electromagnetic wave through a uniform dielectric medium of dielectric constant ϵ. According to Eqs. (810) and (812), the dipole moment per unit volume induced in the medium by the wave electric field $\mathbf{E}$ is

$$\mathbf{P} = \epsilon_0 (\epsilon - 1) \mathbf{E}. \qquad (1139)$$

There are no free charges or free currents in the medium. There is also no bound charge density (since the medium is uniform), and no magnetization current density (since the medium is non-magnetic). However, there is a *polarization current* due to the time-variation of the induced dipole moment per unit volume. According to Eq. (859), this current is given by

$$\mathbf{j} = \frac{\partial \mathbf{P}}{\partial t}. \qquad (1140)$$

Hence, Maxwell's equations take the form

$$\nabla\cdot\mathbf{E} = 0, \tag{1141}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{1142}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{1143}$$

$$\nabla\times\mathbf{B} = \mu_0\,\mathbf{j} + \epsilon_0\,\mu_0\,\frac{\partial\mathbf{E}}{\partial t}. \tag{1144}$$

According to Eqs. (1139) and (1140), the last of the above equations can be rewritten

$$\nabla\times\mathbf{B} = \epsilon_0\,\mu_0\,(\epsilon-1)\,\frac{\partial\mathbf{E}}{\partial t} + \epsilon_0\,\mu_0\,\frac{\partial\mathbf{E}}{\partial t} = \frac{\epsilon}{c^2}\,\frac{\partial\mathbf{E}}{\partial t}, \tag{1145}$$

since $c = (\epsilon_0\,\mu_0)^{-1/2}$. Thus, Maxwell's equations for the propagation of electromagnetic waves through a dielectric medium are the same as Maxwell's equations for the propagation of waves through a vacuum (see Sect. 4.7), except that $c \rightarrow c/n$, where

$$n = \sqrt{\epsilon} \tag{1146}$$

is called the *refractive index* of the medium in question. Hence, we conclude that electromagnetic waves propagate through a dielectric medium *slower* than through a vacuum by a factor n (assuming, of course, that $n > 1$). This conclusion (which was reached long before Maxwell's equations were invented) is the basis of all geometric optics involving refraction.

Poynting theorem and derivation

Statement. This theorem states that the cross product of electric field vector, **E** and magnetic field vector, **H** at any point is a measure of the rate of flow of electromagnetic energy per unit area at that point, that is

P = E x H

Here **P** → Poynting vector and it is named after its discoverer, J.H. Poynting. The direction of **P** is perpendicular to **E** and **H** and in the direction of vector **E x H**

Proof. Consider Maxwell's fourth equation (Modified Ampere's Circuital Law), that is

$$\nabla \times H = J + \varepsilon\, dE/dt$$

or

$$J = (\nabla \times H) - \varepsilon\, dE/dt$$

The above equation has the dimensions of current density. Now, to convert the dimensions into rate of energy flow per unit volume, take dot product of both sides of above equation by **E**, that is

$$E \cdot J = E \cdot (\nabla \times H) - \varepsilon E \cdot dE/dt \qquad (1)$$

Use vector Indentity

$$\nabla \cdot (E \times H) = H \cdot (\nabla \times E) - E \cdot (\nabla \times H)$$

or

$$E \cdot (\nabla \times H) = H \cdot (\nabla \times E) - \nabla \cdot (E \times H)$$

By substituting value of E. (del x **H**) in equation (1) , we get

$$E \cdot J = H \cdot (\nabla \times E) - \nabla \cdot (E \times H) - \varepsilon E\, dE/dt \qquad (2)$$

also from Maxwell's third equation (Faraday's law of electromagnetic induction).

$$\nabla \times E = \mu dH/dt$$

By substituting value of del x E in equation (2) we get

$$E \cdot J = \mu H \cdot (dH)/dt - \varepsilon E \cdot dE/dt - \nabla \cdot (E \times H) \qquad (3)$$

We can write

$$H \cdot dH/dt = 1/2 \; dH^2/dt \qquad (4a)$$

$$E \cdot dE/dt = 1/2 \; dE^2/dt \qquad (4b)$$

By substituting equations 4a and 4b in equation 3 , we get

$$E \cdot J = -\mu/2 \; dH^2/dt - \varepsilon/2 \; dE^2/dt - \nabla \cdot (E \times H)$$

$$E \cdot J = -d/dt \; [dH^2/2 + \varepsilon E^2/2] - \nabla \cdot (E \times H)$$

By taking volume integral on both sides, we get

$$\int E.\,J.\,dV = -d/dt \int [\mu H^2/2 + \varepsilon E^2/2]\, dV - \int del\,.\,(E \times H)\, dV \qquad (5)$$

apply Gauss's Divergence theorem to second term of R.H.S., to change volume integral into surface integral, that is

$$\int del\,.\,(E \times H)\, dV = \int (E \times H)\,.\,dS$$

Substitute above equation in equation 5

Thus

$$\int E.\,J.\,dV = -d/dt \int [\varepsilon E^2/2 + \mu H^2/2]\, dV - \int (E \times H)\,.\,dS \qquad (6)$$

or $$\int (E \times H)\,.\,dS = \int -d/dt\ [\varepsilon E^2/2 + \mu H^2/2]\, dV - \int E.\,J.\,dV$$

Interpretation of above equation :

L.H.S. Term

$\int (E \times H)\,.\,dS$ → It represents the rate of outward flow of energy from a volume **S**

V and the integral is over the closed surface surrounding the volume. This rate of outward flow of power from a volume V is represented by

$$\int P\,.\,dS = \int (E \times H)\,.\,dS$$

where Poynting vector, **P = E x H**

Inward flow of power is represented by

$$-\int P\,.\,dS = -\varphi\int (E \times H)\,.\,ds$$

R.H.S. First Term

$-d/dt\ [\mu H^2/2 + \varepsilon E^2/2]\ dV$ → If the energy is flowing out of the region, there must be a corresponding decrease of electromagnetic energy. So here negative sign indicates decrease. Electromagnetic energy is the sum of magnetic energy, $\mu H^2/2$ and electric

energy, $\varepsilon E^2/2$. So first term of R.H.S. represents rate of decrease of stored electromagnetic 2 energy.

R.H.S. Second Term

∫ **(E. J) dV** →Total ohmic power dissipated within the volume.

So from the law of conservation of energy, equation (6) can be written in words as

Rate of energy disripation in volume V = Rate at which stored electromagnetic energy is decreasing in V + Inward rate of flow of energy through the surface of the volume.

Poynting vector, a quantity describing the magnitude and direction of the flow of energy in electromagnetic waves. It is named after English physicist John Henry Poynting, who introduced it in 1884.

The Poynting vector *S* is defined as to be equal to the cross product (1/μ)*E* × *B*, where μ is the permeability of the medium through which the radiation passes (*see* magnetic permeability), *E* is the amplitude of the electric field, and *B* is the amplitude of the magnetic field. Applying the definition of cross product (*see* vector) and the knowledge that the electric and magnetic fields are perpendicular to each other gives the magnitude *S* of the Poynting vector as (1/μ)*EB*, where *E* and *B* are, respectively, the magnitudes of the vectors *E* and *B*. The direction of the vector product *S* is perpendicular to the plane determined by the vectors *E* and *B*. For a traveling electromagnetic wave, the Poynting vector points in the direction of the propagation of the wave.