Electromagnetic Theory

(A) Maxwell's field equations:

E.M. theory is a subject which deal with electrostatic as well as magnetostatic fields. A static charge can produce electrostatic field whereas a charge in motion can produce magnetostatic field and e.m. theory can be renamed as electrodynamics. An e.m. wave can be characterized by electric field ($\vec{E}$), magnetic field ($\vec{H}$), electric displacement vector ($\vec{D}$) and magnetic induction ($\vec{B}$). These field vectors obey four fundamental equations which are known as Maxwell's field equations. These equations are not absolutely new but extraction from and modification of existing laws in electrostatics and magnetostatics. Let us now derive them.

ⓐ Maxwell's 1st equation $[\vec{\nabla}\cdot\vec{D} = \rho]$:-

Derivation: We have Gauss's theorem in electrostatics which states that "flux of electric field across any closed surface is equal to $1/\epsilon_0$ times the charge enclosed by the surface".

Mathematically, $\oiint \vec{E}\cdot\hat{n}\,ds = \frac{1}{\epsilon_0} q$.

or, $\oiint \vec{E}\cdot\hat{n}\,ds = \frac{1}{\epsilon_0}\left[\iiint_v \rho\,dv - \iiint \vec{\nabla}\cdot\vec{P}\,dv\right]$

where ρ = free vol. charge density

$\rho_p = -\vec{\nabla}\cdot\vec{P}$ = Polarization charge density.

$\vec{P}$ = Polarization having unit C/m^2.

Applying Gauss's divergence theorem in LHS and Rearranging,

$$\iiint_V (\vec{\nabla}\cdot\epsilon_0\vec{E} + \vec{\nabla}\cdot\vec{P})\,dv = \iiint \rho\,dv.$$

or, $$\iiint_V \vec{\nabla}\cdot(\epsilon_0\vec{E} + \vec{P})\,dv = \iiint \rho\,dv.$$

or, $$\iiint (\vec{\nabla}\cdot\vec{D} - \rho)\,dv = 0,$$ where $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ = electric disp. vector.

Since elementary volume dv, is taken arbitrarily, so it can not be zero. Hence above eqn. holds goods if and only if

$$\boxed{\vec{\nabla}\cdot\vec{D} = \rho} \quad \text{------ (1)}$$

Phy. Sig: This is Maxwell's 1st equation.
It is nothing but differential form of Gauss's theorem in electrostm which states that " . . . "

(b) Maxwell's 2nd equation $[\vec{\nabla}\cdot\vec{B} = 0]$

Derivation: Since isolated magnetic poles does not exist, according to Gauss's theorem in magnetostatics, we can write, flux of mag. field over any closed surface is zero i.e

$$\oiint \vec{B}\cdot\hat{n}\,ds = 0 \quad \therefore \boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

Again, $\iiint_V \vec{\nabla}\cdot\vec{B}\,dv = 0$
Since magnetic lines of forces are closed curves / goes off to

infinity, we can say, magnetic ind. is a flux free/solenoidal vector i,e we we can write

$$\vec{\nabla} \cdot \vec{B} = 0$$

This is Maxwell's 2nd eqn.

Physical Significance; HW

© Maxwell's 3rd equation; $\left[\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\right]$

We have Faraday's law of e.m. induction

i,e induced emf $e = -\frac{\partial \psi}{\partial t}$ = time rate of change of mag. flux

Now (4).

emf e = work done to bring unit positive charge once round a closed path in the el. field

$$= \oint_c \vec{F} \cdot d\vec{r} = \oint_c \vec{E} \cdot d\vec{r}$$

and $\psi = \iint_S \vec{B} \cdot \hat{n}\, ds$

$$\therefore \oint_c \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \iint \vec{B} \cdot \hat{n}\, ds.$$

App Stoke's Theorem in LHS,

$$\iint_S (\vec{\nabla} \times \vec{E}) \cdot \hat{n}\, ds = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, ds.$$

or, $$\iint \left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right) \cdot \hat{n}\, ds = 0$$

$$\therefore \boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \quad --(3)$$

This is Maxwell's 3rd eqn. which is nothing but diff. form of Faraday's law in e.m. ind

(d) Maxwell's 4th equation: $\boxed{\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$

[Copy from Part – I]

⟨B⟩ Equation of continuity $\boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0}$

Home work: Copy " from Part-I

⟨C⟩ Constitutive relations :- The following relations are known as Constitute relations :-

i) $\vec{D} = \epsilon . \vec{E}$,

(ii) $\vec{B} = \mu \vec{H}$,

ϵ = Permitivity of the medium $= \epsilon_0 \epsilon_r$

ϵ_0 = Per. of free space $= 8.854 \times 10^{-12}$ F/m

ϵ_r = relative Per. / SIC / dielectric Constant

μ = Permiability $= \mu_0 \mu_r$

$\mu_0 = 4\pi \times 10^{-7}$ H/m.

$\left[\frac{1}{4\pi\epsilon_0}\right] = 9 \times 10^{9}$ m/F.

(iii) ohm's law: $\vec{J} = \sigma \vec{E}$ (H.W / Prove it)

Maxwell's equations for diff. media:

We have Maxwell's field equations:

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}; \quad \vec{J} = \sigma\vec{E}$$

(a) Free Space: Free space is characterized by $\rho = 0$, $\sigma = 0$

$\epsilon = \epsilon_0$, $\mu = \mu_0$

So field equations become;

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \text{——} \; (1)$$

$$\vec{\nabla} \cdot (\mu_0 \vec{H}) = 0$$

or $$\vec{\nabla} \cdot \vec{H} = 0 \quad \text{——} \; (2)$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

(b) Dielectric medium. (c) Conducting medium

V.U/94 5.(b) i) We have Maxwell's 3rd eqn i,e diff. form of Faraday's law;

3rd ⟶ 2nd

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Taking divergence on both sides;

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{E}) = -\vec{\nabla} \cdot \frac{\partial \vec{B}}{\partial t}$$

or, $$0 = -\frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{B})$$

{ Since differentiation w.r.t. time and space are interchangable.

Since $\vec{\nabla} \cdot \vec{B}$ is not independent of time, above eqn. holds good, if $\vec{\nabla} \cdot \vec{B} = 0$

This is Gauss's theorem in mag.. i,e 2nd law.

ii) <u>4th ⟶ 1st</u>

We have modified Ampere's law;

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Taking div. on both sides:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \frac{\partial \vec{D}}{\partial t}$$

or, $$0 = -\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{D})$$

[Since, ... , $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$]

or $$\frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{D} - \rho) = 0$$

$$\therefore \boxed{\vec{\nabla} \cdot \vec{D} = \rho}$$

Since fields are time varying.

This is Gauss's law in electrostatics.

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8.(a) We have Maxwell's 4th eqn:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Taking div. on both sides:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J} + \vec{\nabla} \cdot \frac{\partial \vec{D}}{\partial t}$$

or $$0 = \vec{\nabla} \cdot \vec{J} + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{D})$$, Since diff. form Max. 1st eqn: $\vec{\nabla} \cdot \vec{D} = \rho$

$$\therefore \boxed{\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$