- SEMESTER –II (HONOURS)
- PAPER : C4T (WAVE AND OPTICS)
- TOPIC: VELOCITY OF WAVE

CLASS NOTES :BY TAPAS KUMAR CHANDA

Dept. of physics, Bhatter College,Dantan

March 29, 2020

Subtopics: velocity of transverse and longitudinal wave, Newton's formula , Laplace correction.

Next class: Hermonic Oscillations

X

Velocity of plane progressive wave through elastic solid

V.U — 1997

Let us consider a cylindrical elastic solid medium (b… Let A and B are the two normal planes at a distance x… $x+\delta x$ from some arbitary origin. When the medium… disturbed by applying a force per unit cross section ∂F… let A is displaced to A′ by an amount y while the plane… displaced to B′ by an amount $y + \frac{\partial y}{\partial x}\delta x$.

Original length of the medium between A and B $= \delta$…
and between A′ and B′ $= \delta x$…

Hence increase in length $= \frac{dy}{dx}\delta x$.

$\therefore$ Longitudinal strain $= \frac{\frac{dy}{dx}\delta x}{\delta x} = \frac{dy}{dx}$.

Hence, Young's modulus $= \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$.

$$\Rightarrow Y = \frac{-F}{dy/dx}$$

$$\Rightarrow F = -Y\frac{dy}{dx}.$$

Net force acting between layers A and B

$$= -Y\frac{\partial y}{\partial x}\Big|_{x=x} + Y\frac{\partial y}{\partial x}\Big|_{x=x+\delta x}$$

$$= Y\frac{\partial^2 y}{\partial x^2}\delta x.$$

If ρ be the mass per unit length of the rod, then force = mass × accl.

$$\Rightarrow Y\frac{\partial^2 y}{\partial x^2}\delta x = \rho\,\delta x\cdot\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{\rho}{Y}\frac{\partial^2 y}{\partial t^2} \quad \text{——— (1)}$$

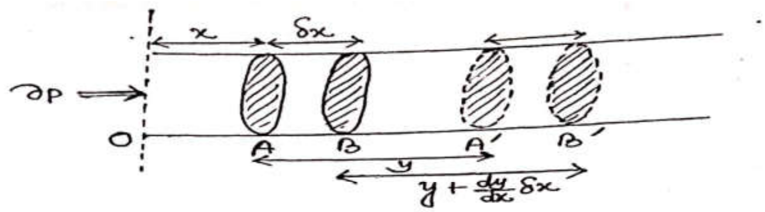
Again one dimensional wave equation,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad \text{——— (2)}$$

Comparing we get, $\boxed{v = \sqrt{\frac{Y}{\rho}}}$

Expression for the velocity of a plane longitudinal wave propagating in gaseous medium or liquid medium :—

We consider a tube of cross-section α in the fluid medium. Let A and B are the two normal plane section at a distance x and $x+\delta x$ from some arbitary origin. When the medium is disturbed by applying an excess pressure ∂p, let A is displaced to A′ by an amount y while the plane B is displaced to B′ by an amount $y+\frac{\partial y}{\partial x}\delta x$.



Original volume of the fluid between plane A and B $= \alpha$

Distance between planes A′ and B′ $= OB' - OA'$

$$= (x+\delta x+y+\frac{dy}{dx}\delta x) -$$

$$= \delta x + \frac{dy}{dx}\delta x.$$

Final volume of the fluid between planes A′ and B′

$$= \alpha(\delta x + \frac{dy}{dx}\delta x)$$

Change in volume of the fluid $= \alpha\delta x + \alpha\frac{dy}{dx}\delta x -$

$$= \frac{dy}{dx}\alpha\delta x$$

Hence volume strain $= \frac{\text{Change in volume}}{\text{Original volume}}$

$$= \frac{\frac{dy}{dx}\alpha\delta x}{\alpha\delta x} = \frac{dy}{dx}.$$

Bulk modulus of the medium $= \frac{\text{Volume stress}}{\text{Volume strain}}$

$$K = -\frac{\delta p}{\left(\frac{dy}{dx}\right)}$$

$$\Rightarrow \boxed{\delta p = -K\frac{dy}{dx}}$$

Net force acting between the layers A and B,

$$= -K\alpha\frac{dy}{dx}\Big|_{x=x} + K\alpha\frac{dy}{dx}\Big|_{x=x+\delta x}$$

$$= -K\alpha\frac{dy}{dx} + K\alpha\frac{dy}{dx} + K\alpha\frac{d^2y}{dx^2}\delta x - - \text{ neglecting higher o}$$

$$= K\alpha\frac{d^2y}{dx^2}\delta x.$$

If ρ be the density of the fluid, then mass of the layer between plane A and B $= \rho \alpha \delta x$.

Hence equation of motion,

$$\text{force} = \text{mass} \times \text{accl.}$$

$$\Rightarrow K\alpha \frac{d^2y}{dx^2} \delta x = \rho \alpha \delta x \times \frac{d^2y}{dt^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \rho/K \frac{d^2y}{dt^2} \quad \text{---} \quad (1)$$

Again one dimensional wave equation,

$$\frac{d^2y}{dx^2} = \frac{1}{v^2} \cdot \frac{d^2y}{dt^2} \quad \text{---} \quad (2)$$

Where v is the velocity of the wave.

Comparing equation (1) and (2), we get —

$$\boxed{v = \sqrt{\frac{K}{\rho}}}$$

Which is the velo. of longitudinal wave through fluid medium.

ii) Newton's law about velo. of sound in gas medium —

According to Newton compression and rarification of gas medium occurs very slowly at the time of propagation of sound wave.

Hence the process is isothermal.

$$\therefore PV = \text{Const}$$

$$\Rightarrow P\,dV + V\,dP = 0$$

$$\Rightarrow -\frac{dP}{dV/V} = P$$

$$\Rightarrow K = P.$$

Hence $\boxed{v = \sqrt{P/\rho}}$

ii) Laplace's correction —

According to Laplace this process occours ver[y] quickly, hence the process is adiabatic.

$\therefore \; PV^{\gamma} = \text{const}$ Where $\gamma = C_p/C_v$

Diff. $P \cdot \gamma V^{\gamma-1} dV + V^{\gamma} dP = 0$.

$\Rightarrow \; -\dfrac{dP}{dV/V} = \gamma P$

$\Rightarrow \; K = \gamma P$.

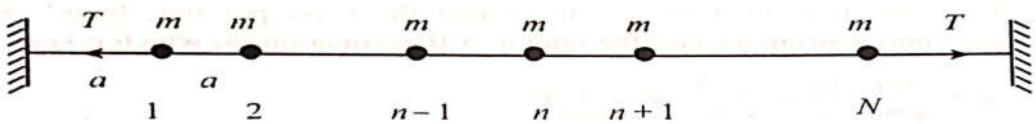
Hence $$v = \sqrt{\frac{\gamma P}{\rho}}$$

A string of negligible mass is fixed at $x = 0$ and $x = l$. The N number of massive beads are fixed on the string at $x = a, 2a, \dots \dots Na$ at equilibrium below figure. Each bead has mass m. The tension of the beaded string is T. Show that the dispersion relation for the normal modes of small transverse oscillations of the beads along the y-direction is

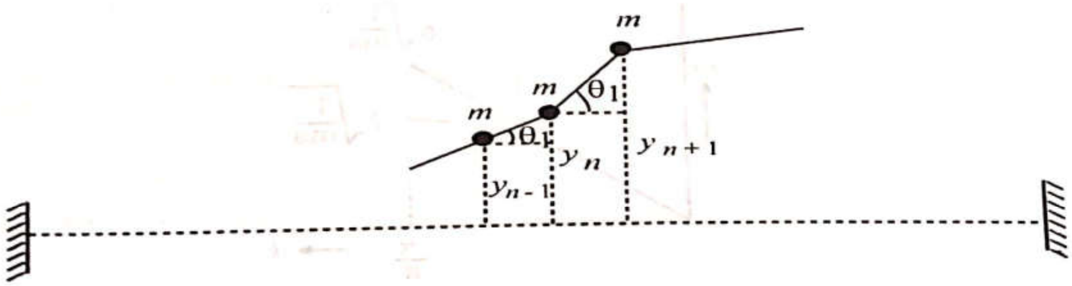
$$\omega = \sqrt{\frac{4T}{ma}} \sin\left(\frac{ka}{2}\right).$$

From this equation, show with proper limiting procedure that the dispersion relation for the massive continuous string is $\omega = \sqrt{\frac{T}{\mu}}\,k$ Where μ is the mass per unit length of the massive continuous string.

In order to find the equation of motion of the nth bead below:



We consider the n-th bead and its neighbours $(n-1)$ (to the left) and $(n+1)$ (to the right). Let y_{n-1}, y_n and y_{n+1} be the transverse displacement of the $(n-1), n$ and $(n+1)$ the bead respectively below:



Then the force on the nth bead along the y-direction is

$$F_y = T\sin\theta_2 - T\sin\theta_1 = T\,\frac{y_{n+1} - y_n}{a} - T\,\frac{y_n - y_{n-1}}{a}$$

Where for small oscillations θ_1 and θ_2 are very small so that we may put $\sin\theta_2 = \tan\theta_2$ and $\sin\theta_1 = \tan\theta_1$.

Thus, $m\frac{d^2y_n}{dt^2} = \frac{T}{a}[y_{n+1} - 2y_n + y_{n-1}]$ ... (1)

For normal modes of oscillations, each bead oscillates harmonically with the same frequency ω and with the same phase constant ϕ:

$y_i = A_i\cos(\omega t + \phi), \quad i = 1, 2, \dots \dots \dots N.$

Now, we have from equation (1) the following difference equation

$$A_{n+1} + A_{n-1} = \left(2 - \frac{m\omega^2 a}{T}\right) A_n. \quad \cdots (2)$$

Let us try a solution of equation (2) in the form

$$A_n = B \sin(kna) \quad \cdots (3)$$

$$\sin[k(n+1)a] + \sin(k(n-1)a] = \left(2 - \frac{m\omega^2 a}{T}\right) \sin(kna)$$

$$\text{Or, } 2\sin(kna)\cos ka = \left(2 - \frac{m\omega^2 a}{T}\right) \sin(kna)$$

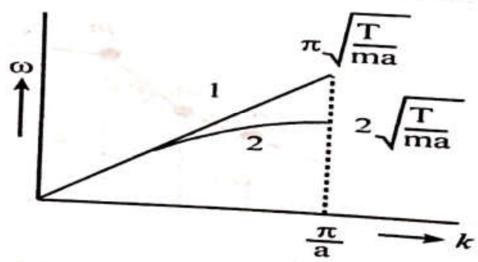
Hence equation (3) is a solution provided $2\cos ka = 2 - \frac{m\omega^2 a}{T}$ Or, $\omega^2 = \frac{4T}{ma}\sin^2\left(\frac{ka}{2}\right)$.

Thus, the required dispersion relation is $\omega = \sqrt{\frac{4T}{ma}}\sin\left(\frac{ka}{2}\right)$. ... (4)

In a length 'a' total mass $= m$ so that the mass per unit length $= \mu = m/a$. For the continuous string we take the limit $a \to 0$ in equation (4) which gives

$$\omega = \sqrt{\frac{4T}{ma}} \cdot \frac{ka}{2} = \sqrt{\frac{Ta^2}{ma}} \cdot k = \sqrt{\frac{T}{\mu}}\, k \quad \ldots (5)$$

The $\omega - k$ graph of the continuous string (1) and the beaded string (2) is shown in fig. below:

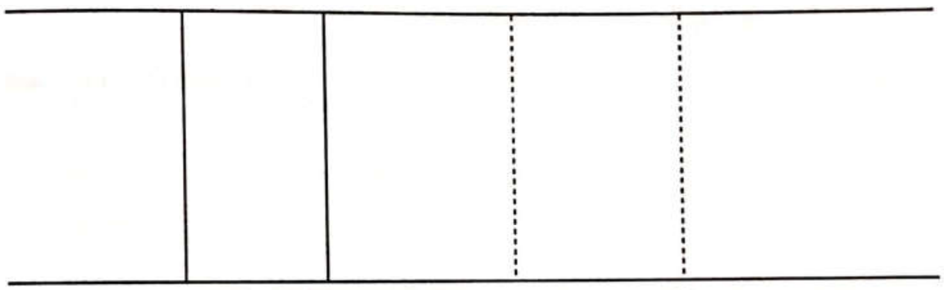


4.1.1. Show that the velocity of propagation of longitudinal waves in a fluid (liquid or gas) contained in an infinitely long tube is given by $v = \sqrt{K/\rho_0}$

Where k = Bluk modulus of the fluid,

ρ_0 = Equilibrium density of the fluid.

Ans: We consider an infinitely long tube of cross section A (Fig below), containing a fluid (liquid or gas). Suppose that originally the fluid is at rest, its density is ρ_0 and pressure P_0 Let R be a section of the medium of area A, perpendicular to the direction of propagation of the wave. Let S be a parallel section of equal area δx apart, δx being an elementary thickness of the layer. The co-ordinates of R and S are x and $x + \delta x$ respectively. Originally the small cylinder of fluid RS experiences an equal pressure P_0 exerted at both ends by the



surrounding fluid. Suppose the fluid is set into longitudinal agitation, for example by inserting a piston in the tube somewhere to the left of R and causing it to oscillate longitudinally. This will set the adjacent fluid into longitudinal oscillation, and this disturbance will propagate along the fluid in the form of a longitudinal wave. Suppose that at a given instant, the cylinder RS is displaced to a new position $R'S'$ such that $R'R = \xi$ and $S'S = \xi + \frac{\partial \xi}{\partial x}\delta x$. The variable ξ measures the longitudinal displacement of a point due to the passage of the wave. The thickness of the layer $R'S'$ is $\delta x + \frac{\partial \xi}{\partial x}\delta x$. Thus, the increase of thickness of the layer due to the longitudinal oscillation at that instant $\frac{\partial \xi}{\partial x}\delta x$. Since there is no motion perpendicular to the direction of propagation of the wave, the corresponding increase in volume $= A\frac{\partial \xi}{\partial x}\delta x$.

We shall find the equation of motion of the fluid at $R'S''$. For this purpose we require to know its mass and the pressure at its two ends. Its mass is same as the mass of the undisturbed element RS, that is $A\rho_0\delta x$, where δ_0 = normal average density or equilibrium density of the fluid. Let the pressure on the left-hand face R' be P and that on the right-hand face S' be $P + \delta P$. The bluk modulus K of a material is a measure of the pressure increase to change its volume by a given amount. It is defined as

$$K = \frac{\text{Stress}}{\text{Strain}} = \frac{\text{Extra pressure applied}}{\text{Fractional change in volume}}.$$

Proceeding to the limit of vanishing small change in volume, we obtain

$$K = -\frac{dP}{\frac{dV}{V}} = -V\frac{dP}{dV}.$$

The minus sign indicates that the volume decreases if the pressure increases

$$\text{Volume strain} = \frac{dV}{V} = \frac{\text{increase in volume}}{\text{original volume}} = \frac{A\frac{\partial \xi}{\partial x}\delta x}{A\delta x} = \frac{\partial \xi}{\partial x}$$

Thus from equation (1) we find that the extra pressure over normal undisturbed pressure

$$= -K\frac{dV}{V} = -K\frac{\partial \xi}{\partial x}$$

If we define acoustic pressure as $p = P - P_0$

We have $p = -K\frac{\partial \xi}{\partial x}$.

Equating the net force on the displaced cylinder $R'S'$ to the product of mass and acceleration (Newton's second law), we get

$$PA - (P + \delta P)A = (A\rho_0\delta x)\frac{\partial^2 \xi}{\partial t^2}$$

Or, $-\delta P = \rho_0\delta x\frac{\partial^2 \xi}{\partial t^2}$

From equation (3) we have $\delta p = \delta P$ since P_0 is constant. Thus,

$$-\delta p = -\frac{\partial p}{\partial x}\delta x = (\rho_0\delta x)\frac{\partial^2 \xi}{\partial t^2}$$

Using equation (4) we obtain $K\frac{\partial^2 \xi}{\partial x^2} = \rho_0\frac{\partial^2 \xi}{\partial t^2}$

Or, $\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{K/\rho_0}\frac{\partial^2 \xi}{\partial t^2}$ ...(6)

$v = \sqrt{K/\rho_0}$ ...(7)

4.1.2. In the previous problem show that $\rho = \rho_0\left(1 - \frac{\partial \xi}{\partial x}\right)$

Where ρ_0 = equilibrium density of the fluid

ρ = density fo the fluid in the disturbed position and $\frac{\partial \xi}{\partial x}$ = volume strain.

Ans: Volume of the fluid at $R'S' = A\delta x + A\frac{\partial \xi}{\partial x}\delta x$.

Hence, $\rho = \frac{A\rho_0\delta x}{A\delta x + A\frac{\partial \xi}{\partial x}\delta x} = \frac{\rho_0}{1+\frac{\partial \xi}{\partial x}} = \rho_0\left(1 - \frac{\partial \xi}{\partial x}\right)$

Where $\frac{\partial \xi}{\partial x} \ll 1$.

Note: If we define condensation (s) as

$s = \frac{\rho - \rho_0}{\rho_0}$, then we find $s = -\frac{\partial \xi}{\partial x}$ and $\rho = \rho_0(1 + s)$