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Topic - Waveguide

Wave guides:= Wave guide is a hollow pipe of infinite extent.

We shall now consider the propagation of electro-magnetic waves along a hollow conducting pipe of arbitrary cross section uniform along its length.

We shall assume that the walls of the wave guide are perfectly conducting and take the Z axis along the wave guide.

For simplicity we further assume that the interior of wave guide is vaccum. Each component of E and B must satisfy wave equation in vaccum has the form

$$\nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = 0$$

$$\text{or, } \nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

Again,
$$\nabla^2 B - \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2} = 0$$

$$\text{or, } \nabla^2 B - \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} = 0$$

vacuum. As shown in chapter 7, each component of **E** and **B** must satisfy wave equation which in vacuum has the form

$$\left.\begin{aligned}\nabla^2\mathbf{E}-\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}&=0 \text{ or } \nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0 \quad ...(a)\\ \nabla^2\mathbf{B}-\mu_0\varepsilon_0\frac{\partial^1\mathbf{B}}{\partial t^2}&=0 \text{ or } \nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=0 \quad ...(b)\end{aligned}\right\} \quad ...(1)$$

In addition to these wave equations Maxwell's equations and boundary conditions must be satisfied. As the wave is propagated along Z-axis, we assume solutions of the form

$$\left.\begin{aligned}\mathbf{E}(\mathbf{r},t)&=\mathbf{E}(x,y)\,e^{ik_g z-i\omega t} \quad ...(a)\\ \mathbf{B}(\mathbf{r},t)&=\mathbf{B}(x,y)\,e^{ik_g z-i\omega t} \quad ...(b)\end{aligned}\right\} \quad ...(2)$$

Appropriate linear combinations can be formed to give travelling or standing waves in the Z-direction. The wave number $k_g$ is known from preceding section but however we assume that it is unknown parameter which may be real or complex. With this assumed Z dependence of fields, the wave equation reduces to two dimensional form

$$\frac{\partial^2\mathbf{E}}{\partial x^2}+\frac{\partial^2\mathbf{E}}{\partial y^2}-k_g^2\mathbf{E}+\frac{\omega^2}{c^2}\mathbf{E}=0$$

$$\left.\begin{aligned}\text{or}\quad \frac{\partial^2\mathbf{E}}{\partial x^2}+\frac{\partial^2\mathbf{E}}{\partial y^2}+\left[\frac{\omega^2}{c^2}-k_g^2\right]\mathbf{E}&=0 \quad ...(a)\\ \text{and}\quad \frac{\partial^2\mathbf{B}}{\partial x^2}+\frac{\partial^2\mathbf{B}}{\partial y^2}+\left[\frac{\omega^2}{c^2}-k_g^2\right]\mathbf{B}&=0 \quad ...(b)\end{aligned}\right\} \quad ...(3)$$

Now if $\nabla_\perp^2$ is transverse part of Laplacing operator $\nabla^2$, then

$$\nabla_\perp^2=\nabla^2-\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \quad ...(4)$$

In these notations equation (3) takes the form

$$\left[\nabla_\perp^2+\left(\frac{\omega^2}{c^2}-k_g^2\right)\right]\begin{Bmatrix}\mathbf{E}\\ \mathbf{B}\end{Bmatrix}=0 \quad ...(5)$$

Now Maxwell's curl equations for free space are

$$\left.\begin{aligned}\nabla\times\mathbf{E}&=-\frac{\partial\mathbf{B}}{\partial t} \quad (a)\\ \text{and}\quad \nabla\times\mathbf{B}&=\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \text{ or } \nabla\times\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \quad ...(b)\end{aligned}\right\} \quad ...(6)$$

which in terms of components can be written as

$$\left.\begin{aligned}\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}&=-\frac{\partial B_x}{\partial t}\\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}&=-\frac{\partial B_y}{\partial t}\\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}&=-\frac{\partial B_z}{\partial t}\end{aligned}\right] \quad ...(7)$$

and

$$\left.\begin{aligned}\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}&=\frac{1}{c^2}\frac{\partial E_x}{\partial t}\\ \frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}&=\frac{1}{c^2}\frac{\partial E_y}{\partial t}\\ \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}&=\frac{1}{c^2}\frac{\partial E_z}{\partial t}\end{aligned}\right]$$

From the form of exponential $e^{ik_g z - i\omega t}$ it is apparent that

$$\frac{\partial}{\partial z}\to ik_g \text{ and } \frac{\partial}{\partial t}\to -i\omega$$

Therefore equations (7) lead to

$$\left.\begin{aligned}\frac{\partial E_z}{\partial y}-ik_g E_y&= i\omega B_x & \text{....}(a)\\ ik_g E_x-\frac{\partial E_z}{\partial y}&= i\omega B_z & \text{....}(b)\\ \frac{\partial \mathbf{E_y}}{\partial x}-\frac{\partial \mathbf{E_x}}{\partial y}&= i\omega B_z & \text{...}(c)\end{aligned}\right]$$

and

$$\left.\begin{aligned}\frac{\partial B_z}{\partial y}-ik_g B_y&= -\frac{i\omega}{c^2}\mathbf{E_x} & \text{....}(a)\\ ik_g B_x-\frac{\partial B_z}{\partial x}&= -\frac{i\omega}{c^2}\mathbf{E_y} & \text{....}(b)\\ \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}&= -\frac{i\omega}{c^2}\mathbf{E_z} & \text{...}(c)\end{aligned}\right]$$

Substituting values of $B_y$ from (8b) in (9a), we get

$$\frac{\partial B_z}{\partial y}-ik_g\left(\frac{ik_g\mathbf{E_x}-\dfrac{\partial \mathbf{E_z}}{\partial x}}{i\omega}\right)=-\frac{i\omega}{c^2}\mathbf{E_x}$$

*i.e.*

$$\frac{\partial B_z}{\partial y}+\frac{k_g}{\omega}\frac{\partial \mathbf{E_z}}{\partial x}=\left(\frac{ik_g^2}{\omega}-\frac{i\omega}{c^2}\right)\mathbf{E_x}$$

*i.e.*

$$\mathbf{E_x}=\frac{i}{\left(\dfrac{\omega^2}{c^2}-k_g^2\right)}\left[k_g\frac{\partial \mathbf{E_z}}{\partial x}-\omega\frac{\partial B_y}{\partial y}\right]$$

Now eliminating $B_x$ from (8a) and (9b), we get

$$\mathbf{E_y}=\frac{i}{\left[\dfrac{\omega^2}{c^2}-k_g^2\right]}\left[k_g\frac{\partial \mathbf{E_z}}{\partial q}-\omega\frac{\partial B_y}{\partial x}\right]$$

Similarly eliminating $\mathbf{E}_y$ and $\mathbf{E}_x$ in turn from (8) and (9), we get

$$B_x = \frac{i}{\left(\frac{\omega^2}{c^2} - k_g^2\right)}\left(k_g \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial \mathbf{E_z}}{\partial y}\right) \quad ...(12)$$

$$B_y = \frac{i}{\left(\frac{\omega^2}{c^2} - k_g^2\right)}\left(k_g \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial \mathbf{E_z}}{\partial x}\right) \quad ...(13)$$

Equation (10), (11), (12) and (13) show that it is sufficient to determine $\mathbf{E_z}$ and $B_z$ as the appropriate solutions of the two dimensional wave equation (5). The other components can then be calculated from (10), (11), (12) and (13). In general cylindrical guide with perfectly conducting walls, such as that shown in fig. 9.11, is under consideration, then the appropriate boundary conditions are that the tangential component of **E** and the normal component of **B** should vanish on the surface of the conductor. The boundary condition for two dimensional wave equation for $\mathbf{E}_z$ viz.

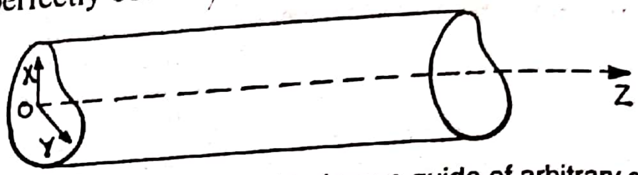

Fig. 9.11 Hollow cylindrical wave guide of arbitrary cross sectional shape

$$\nabla_\perp^2 \mathbf{E_z} + \left(\frac{\omega^2}{c^2} - k_g^2\right)\mathbf{E}_z = 0 \quad ...(14)$$

that the tangential component of **E** should vanish on the surface of the conductor requires

$$\mathbf{B}_z \big|_S = 0 \quad ...(15)$$

The normal component of **B** at the surface requires

$$|\mathbf{B} \cdot \mathbf{n}|_S = 0$$

where **n** is unit vector normal the surface

*i.e.*
$$\left| B, (x, y)\, e^{ik_g z - i\omega t} \cdot \mathbf{n} \right|_S = 0$$

this condition implies
$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0 \quad ...(16)$$

where $\partial/\partial x$ is the normal derivative at the point on the surface. The two dimensional wave equations (5) for $\mathbf{E_z}$ and $B_z$ together with the boundary conditions on $\mathbf{E_z}$ and $B_z$ at the surface of the cylinder allow only certain values of axial wave number $k_g$ for a given frequency ω. Because of different boundary conditions on $\mathbf{E_z}$ and $B_z$ they can not be generally satisfied simultaneously. Consequently the fields may be divided into two distinct classes.

**Transverse Magnetic (TM) Mode.** In this case $B_z = 0$ always *i.e.* magnetic fields is always perpendicular to the direction of propagation; hence the name transverse magnetic. These type of wave are also sometimes named as *electric type waves* or **E** waves.

Substituting $B_z = 0$, equations (10) to (13) take to form

$$\mathbf{E_x} = \frac{ik_g}{\left(\frac{\omega^2}{c^2} - k_g^2\right)}\frac{\partial \mathbf{E_z}}{\partial x} = \frac{ik_g}{(k_o^2 - k_g^2)}\frac{\partial \mathbf{E_z}}{\partial x} \quad ...(17)$$

$$\mathbf{E}_y = \frac{ik_g}{\left(\frac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial \mathbf{E}_z}{\partial y} = \frac{ik_g}{(k_o^2 - k_g^2)} \frac{\partial \mathbf{E}_z}{\partial y} \quad ...(18)$$

$$\mathbf{B}_x = -\frac{i\omega}{\left(\frac{\omega^2}{c^2} - k_g^2\right) c^2} \frac{\partial \mathbf{E}_z}{\partial y} = -\frac{i\omega}{(k_o^2 - k_g^2)\, c^2} \frac{\partial \mathbf{E}_z}{\partial y} \quad ...(19)$$

$$\mathbf{B}_y = \frac{i\omega}{\left(\frac{\omega^2}{c^2} - k_g^2\right) c^2} \frac{\partial \mathbf{E}_z}{\partial y} = \frac{i\omega}{(k_o^2 - k_g^2)\, c^2} \frac{\partial \mathbf{E}_z}{\partial x} \quad ...(20)$$

where $k_0 = \frac{\omega}{c}$ = free space wave number

Thus in *TM* mode, all the transverse components of **E** and **B** can be expressed in terms of longitudinal component of the electric field. This component may be obtained by solving the two dimensional wave equation (5) for $\mathbf{E}_z$ *viz*.

$$\nabla_\perp^2 \mathbf{E}_z + \left(\frac{\omega^2}{c^2} - k_g^2\right) \mathbf{E}_z = 0 \quad ...(21)$$

or

$$\nabla_\perp^2 \mathbf{E}_z + (k_0^2 - k_g^2)\, \mathbf{E}_z = 0$$

The boundary condition for this equation is $\mathbf{E}_z |_S = 0$ ...(i)

**(ii) Transverse electric (TE) Mode.** In this case $\mathbf{E}_z = 0$ always *i.e.* electric field is always perpendicular to the direction of propagation of the wave, hence the name *transverse electric mode*. These types of waves are also called magnetic type waves or *H*-waves.

Substituting $\mathbf{E}_z = 0$, equations (10) to (13) take the form

$$\mathbf{E}_x = \frac{i\omega}{\left(\frac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial \mathbf{B}_z}{\partial y} = \frac{i\omega}{k_0^2 - k_g^2} \frac{\partial \mathbf{B}_z}{\partial y} \quad ...(22)$$

$$\mathbf{E}_y = -\frac{i\omega}{\left(\frac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial \mathbf{B}_z}{\partial x} = -\frac{i\omega}{k_0^2 - k_g^2} \frac{\partial \mathbf{B}_z}{\partial y} \quad ...(23)$$

$$B_z = \frac{ik_g}{\left(\frac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial B_z}{\partial x} = -\frac{ik_g}{k_0^2 - k_g^2} \frac{\partial B_z}{\partial x} \quad ...(24)$$

$$B_y = \frac{ik_g}{\left(\frac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial B_z}{\partial y} = -\frac{ik_g}{k_0^2 - k_g^2} \frac{\partial B_z}{\partial y} \quad ...(25)$$

Thus in *TE* mode all transverse components of **E** and **B** can be expressed in terms of longitudinal component of magnetic field. This component may be obtained by solving two dimensional wave equation (5) for $B_z$ *viz*.

$$\nabla_\perp^2 B_z + \left(\frac{\omega^2}{c^2} - k_g^2\right) B_z = 0$$

or

$$\nabla_\perp^2 B_z + (k_0^2 - k_g^2) B_z = 0 \quad ...(26)$$

with the boundary condition

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0 \quad ...(i)$$

According to equations (22)–(25), this condition ensures that normal component of **B** is zero.

Thus the problem of determining the electromagnetic field in a wave-guide reduces to that of finding solutions of two dimensional wave equation

$$[\nabla_\perp^2 + (k_o^2 - k_g^2)]\,\psi = 0 \text{ or } [\nabla_\perp^2 + k_c^2]\,\psi = 0 \quad ...(27)$$

subject to boundary conditions $\psi|_S = 0$ and $\partial\psi/\partial n\,|_S = 0$.

Here $$k_c^2 = k_0^2 - k_g^2 \text{ or } k_0^2 = k_c^2 + k_g^2 \quad ...(28a)$$

Equation (28 a) is equivalent to $$\frac{1}{\lambda_g^2} = \frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} \quad ...(28\ b)$$

which is same as equation (11) of preceding section 9.11 Thus the conclusions drawn in the preceding section are quite general and are not limited to the case of propagation between parallel conducting planes.

For a given cross section the solution $\psi$ exists only for a certain definite eigen values of the parameter $k_c^2$. It is easy to see that the constant $k_c^2$ must be non-negative, since roughly speaking $\psi$ must be oscillatory in order to satisfy required boundary condition on opposite sides of the cylinder. These will from a spectrum of eigen values $(k_c^2)_\lambda$ and corresponding solutions $\psi_\lambda$ ($\lambda = 1, 2, 3$) form an orthogonal set. These different solutions are called the *modes of the guide*. For a given frequency $\omega$ the wave number $k_g$ is determined for each value of $\lambda$

$$(k_g^2)_\lambda = k_0^2 - (k_c^2)_\lambda = \omega^2/c^2 - (k_c^2)_\lambda \quad ...(29)$$

Since $k_g^2 = k_0^2 - k_c^2$, therefore we note that for $k_0 > k_c$ or $\omega > \omega_c$, the wave number $k_g$ is real ; hence waves of such modes can propagate in the guide. But if $k_c < k_c$ or $\omega < \omega_c$, $k_g$ is imaginary which is turn implies the attenuation of **E** and **B** given by (2) : hence such modes can not propagate and are called *cut off modes*. The frequency given by

$$(\omega_c)\lambda = [c]\,(k_c)$$

is called cut off frequency. Then the *wave number* can be written ...(30)

as

$$(k_g)\lambda = \frac{1}{c}\sqrt{[\omega^2 - (\omega_c^2)\,\lambda]} \quad ...(31)$$

From this it follows that a guide acts a sort of high pass filter in the sense that only frequencies greater that cut off frequency can be propagated in the guide. Moreover at any given frequency only a finite mumber of modes can propagate. It is often convenient to choose the dimensions of the guide so that at the operating frequency only the lowest mode can occur.

**The velocity of propagation** *of the wave (i.e. group velocity) along the wave guide* is given by the derivative

$$v_z = \frac{\partial\omega}{\partial k_g} = \frac{\partial}{\partial k_g} c\,(k_g^2 + k_c^2)^{1/2}$$

$$= \frac{ck_g}{\sqrt{(k_0^2 + k_c^2)}} = \frac{ck_g}{k_0} = \frac{c^2 k_g}{\omega} \quad ...(32)$$

For a given $k_c$ : this varies from $o$ to $c$ when $k_g$ varies from $o$ to $\infty$.

The **phase velocity** $v_g$ in the guide is given by

$$v_g = \frac{\omega}{k_g} = \frac{\omega}{\sqrt{(\omega^2 - \omega_c^2)}/c} = \frac{c}{\sqrt{\left[1 - \left(\frac{\omega_c^2}{\omega^2}\right)\right]}}$$

Obviously $v_g > c$ and becomes infinite exactly at cut off frequency. The energy in the guide is propagated with the group velocity.

***TEM* Waves.** In addition to *TE* and *TM* modes, there is a degenerate mode, called the transverse electromagnetic (*TEM*) mode, in which both $E_z$ and $B_z$ vanish. If we substitute $E_z = B_z = 0$ in equations (10)–(13), we note that there is no non-zero component of **E** or **B**. This implies that *TEM* **wave can not be propagated along the wave-guide.**

## 9.13. Rectangular Wave Guide

The most commonly used wave guide is that of rectangular cross-section having inner dimension $a$ and $b$ as shown in fig. 9.21.

The solution of two dimensional wave equation

$$(\nabla_\perp^2 \neq k_c^2)\, \psi = 0 \quad ...(1)$$

can be carried out in rectangular coordinates as follows :

***TE* Mode.** For *TE* mode $E_z = 0$ : hence equation (1) is to be written for $B_z$ ; which takes the form

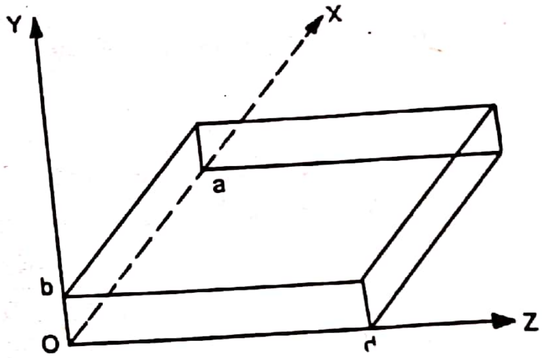


Fig. 9.12

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) B_z = 0$$

The boundary conditions are

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0$$

i.e.

$$\frac{\partial B_z}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$

and

$$\frac{\partial B_z}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b$$

We shall solve equation (2) by the method of separation of variables.

Therefore writing

$$B_z(x, y) = X(x)\, Y(y) = XY$$

where $X$ is a function of $x$ only and $Y$ is a function of $y$ only.

Susbtituting equation (3) in (2) and dividing by $XY$, we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_c^2 = 0$$

or $$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_c^2 = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

In above equation L.H.S. is a function of $x$ only, while R.H.S. is a function of $y$ only. Hence this equation will be satisfied if both sides are equal to a constant say $p^2$ *i.e.*

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + k_c^2 = +p^2$$

or $$\frac{\partial^2 X}{\partial x^2} + (k_c^2 - p^2) X = 0$$

or $$\frac{\partial^2 X}{\partial x^2} + q^2 X = 0 \quad \text{...(4)}$$

where $$q^2 = k_c^2 - p^2 \quad \text{...(5)}$$

and $$-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = p^2$$

or $$\frac{\partial^2 Y}{\partial y^2} + p^2 Y = 0 \quad \text{...(6)}$$

The solution of equation (4) and (6) are

$$X(x) = A\cos qx + B\sin qx \quad \text{...(7)}$$
$$Y(y) = C\cos py + D\sin py \quad \text{...(8)}$$

where $A, B, C, D$ are arbitrary constants.

We have the boundary conditions

$$\frac{\partial B_z}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$
$$\frac{\partial B_z}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b$$

These conditions are equivalent to

$$\frac{\partial X}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$

and $$\frac{\partial Y}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b.$$

Differentiating equations (7) and (8), we get

$$\frac{\partial X}{\partial x} = -Aq\sin qx + Bq\cos qx$$
$$\frac{\partial Y}{\partial y} = -Cp\sin qy + Dp\cos py$$

Applying boundary condition $\left.\frac{\partial Z}{\partial x}\right|_{x=a} = 0$, we get

$Bq = 0 \therefore$ This gives $B = 0$

Now applying boundary condition $\left.\frac{\partial X}{\partial x}\right|_{x=a} = 0$, we get

$$-Aq\sin qa = 0$$

We must take $A \neq 0$ since otherwise $X = 0$ and $B_z = 0$. Hence

$$\sin qa = 0 \text{ or } qa = m\pi, \text{ that is,}$$

$$q = \frac{m\pi}{a} \quad (m \text{ integer}) \qquad ...(11)$$

In precisely the same manner we conclude that $D = 0$ and $p$ must be restricted to values $p = n\pi/b$ where $n$ is an integer. In this way be obtain the solutions

$$X(x) = A\cos\left(\frac{m\pi}{a}\right)x\,;\quad Y(y) = C\cos\left(\frac{n\pi}{b}\right)y \qquad ...(12a)$$

where

$$m = 1, 2, 3, \ldots\ldots \quad n = 1, 2, 3, \ldots.$$

and

$$k_c^2 = p^2 + q^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

The solution for $B_z(x, y)$ is consequently

$$B_z(x, y) = B_0 \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \qquad ...(13a)$$

when

$$(k_c^2)_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2} \qquad ...(13b)$$

Here the indices $mn$ specify the mode. The *cut off frequency* $\omega_{mn}$ is given by

$$\omega_{mn} = \pi c\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^{1/2} \qquad ...(14)$$

The modes corresponding to $m$ and $n$ are represented as $TE_{mn}$. The case $m = n = 0$ gives a static field which does not represent a wave propagation ; hence the mode $TE_{00}$ represents non-trivial solution. If $a > b$, the lowest *cut off frequency* results for $m = 1$ and $n = 0$,

i.e.

$$\omega_{10} = \frac{\pi c}{a} \quad \text{or} \quad (k_c)_{10} = \frac{\pi}{a} \qquad ...(15)$$

The mode ($TE_{10}$) represents the *dominant TE mode* and is the one used in most practical situations. The values $E_x$, $E_y$, $B_x$ and $B_y$ for *TE* mode may be obtained from equations (22)–(25) of preceding section by substituting the solution for $B_z$, which is

$$B_z(r, t) = B_z(x, y)\, e^{ik_g z - i\omega t}$$

$$= B_0 \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}\, e^{ik_g z - i\omega t} \qquad ...(16)$$

Thus we have

$$\left.\begin{aligned} E_x &= \frac{in\pi\omega}{k_c^2 b} B_0 \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{ik_g z - i\omega t} \\ E_y &= \frac{im\pi\omega}{k_c^2 a} B_0 \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{ik_g z - i\omega t} \\ B_x &= -\frac{im\pi k_g}{k_c^2 a} B_0 \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{ik_g z - i\omega t} \\ B_y &= -\frac{in\pi k_g}{k_c^2 a} B_0 \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{ik_g z - i\omega t} \end{aligned}\right\} \quad ...(17)$$

For *TE* mode, these equations yield

$$\left(\text{put } m = 1, n = 0, k_c^2 = \frac{\pi^2}{a^2}\right)$$

$$\left.\begin{aligned} E_x = B_y = E_z = 0;\ B_z &= B_0 \cos\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t} \\ \text{and} \quad B_x &= -\frac{ik_g a}{a} B_0 \sin\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t} \\ E_y &= \frac{i\omega a}{\pi c} B_0 \sin\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t} \end{aligned}\right\} \quad ...(18)$$

The presence of a factor $i$ in $B_x$ (and $E_y$) means that there is a spatial (or temporal) phase difference of $\pi/2$ between $B_x$ (and $E_y$) and $B_z$ in the propagation.

***TM* Mode :** For *TM* mode $B_z = 0$; hence equation (1) is to be written for $E_z$ which takes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) E_z = 0 \quad ...(19)$$

The boundary conditions are $E_z = 0$ at $x = 0, x = a, y = 0$ and $y = b$.

Solving equation (19) as for *TE* case, we note that the solution of equation (19) is of the form

$$E_z(x, y) = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad ...(2$$

where $k_c^2$ and hence $\omega_{mn}$ are still given by equations (12b) and (14). This implies that *TE and TM modes o rectangular guide have the same set of cut off frequencies.* However in this case $m = 1$ and $n = 0$ represen non-trival solution since this gives $E_z = 0$ and hence all components of **E** and **B** will be zero.

It is obvious that in this case the lowest mode has $m = n = 1$ and may be represented by $TM_{11}$. The c **off frequency of lowest mode** is given by

$$\omega_{11} = \pi c \left[\frac{1}{a^2} + \frac{1}{b^2}\right]^{1/2} = \frac{\pi c}{a}\left[1 + \frac{a^2}{b^2}\right]^{1/2} \quad ...($$

Since $a < b$, therefore the *cut off frequency of lowest TM mode is greater than that of the lowest TE mod the factor* $\left[1 + \left(\frac{a^2}{b^2}\right)\right]^{1/2}$ The fields $E_x$, $E_y$, $B_x$, $B_y$ for *TM* mode may be obtained from equat (17)–(22) of preceding section if we substitute

$$E_z(\mathbf{r}, t) = E_0(x, y)\, e^{ik_g z - i\omega t}$$

or
$$E_z(\mathbf{r}, t) = E_0 \sin\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{-ik_g z - i\omega t}$$

Thus we have

$$E_x = \frac{im\pi k_g}{k_c^2 a} E_0 \cos\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t}$$

$$E_y = \frac{in\pi k_g}{k_c^2 b} E_0 \sin\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t}$$

$$B_x = -\frac{i\omega n\pi}{k_c^2 bc} E_0 \sin\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t}$$

$$B_y = \frac{i\omega n\pi}{k_c^2 ac^2} E_0 \cos\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t}$$

## 9.14. Circular Wave Guide

The next most important example of wave guides after the rectangular guide is that in which the bounding surfaces are circular cylinders. This includes both the circular guide and the coaxial line, which has two concentric cylinder, with the wave propagated in the angular space between them. In either case the two dimensional wave equation *viz.*

$$(\nabla_\perp^2 + k_c^2)\,\psi = 0 \qquad \ldots(1)$$

is to be solved in cylinderical coordinates, but here we shall restrict ourselves for cases where cross-section is circular. Let a be the radius of circular cross-section

We have

$$\nabla_\perp^2 = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]$$

Therefore two dimensional wave equation (1) in this case becomes

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]\psi + k_c^2\,\psi = 0 \qquad \ldots(2)$$

If $a$ is the radius of circular cross-section, then the boundary conditions are

$$\psi\Big|_{r=a} = 0 \text{ and } \frac{\partial \psi}{\partial n}\Big|_{r=a} = 0 \quad \text{for all values of } \theta \qquad \ldots(3)$$

Now let us write Maxwell's equations in cylinderical coordinates $(r, \theta, z)$. Maxwell's curl equations in free space are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \ldots(4)$$

In cylindrical coordinates equations take the form

$$\left(\frac{1}{r}\frac{\partial E_r}{\partial \theta} - \frac{\partial E_\theta}{\partial z}\right)\hat{n}_r + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right)\hat{n}_r + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,E_\theta) - \frac{\partial E_r}{\partial \theta}\right]\hat{n}$$

$$= -\left[\frac{\partial B_r}{\partial t}\hat{n}_r + \frac{\partial B_\theta}{\partial t}\hat{n}_\theta + \frac{\partial B_z}{\partial t}\hat{n}_z\right]$$

and

$$\left[\frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial r}\right]\hat{n}_r + \left[\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right]\hat{n}_r + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,B_\theta) - \frac{\partial B_r}{\partial \theta}\right]\hat{n}_z$$

$$= \frac{1}{c^2}\left[\frac{\partial E_r}{\partial t}\hat{n}_r + \frac{\partial E_\theta}{\partial t}\hat{n}_\theta + \frac{\partial E_z}{\partial t}\hat{n}_z\right]$$

Comparing coefficients of $\hat{n}_r$, $\hat{n}_\theta$ and $\hat{n}_z$ on either sides, we get

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial E_z}{\partial r} - \frac{\partial E_\theta}{\partial r} &= -\frac{\partial B_r}{\partial t} \\ \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} &= -\frac{\partial B_\theta}{\partial t} \\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r E_\theta) - \frac{\partial E_r}{\partial \theta}\right] &= -\frac{\partial B_z}{\partial t} \end{aligned}\right] \text{ and}$$

and

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} &= \frac{1}{c^2}\frac{\partial E_r}{\partial t} \\ \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} &= \frac{1}{c^2}\frac{\partial E_\theta}{\partial t} \\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r B_\theta) - \frac{\partial B_r}{\partial \theta}\right] &= \frac{1}{c^2}\frac{\partial E_z}{\partial t} \end{aligned}\right] \quad \text{...(5)}$$

But

$$\mathbf{E}(r, \theta, z, t) = \mathbf{E}(r, \theta)\, e^{ik_g z - i\omega t}$$

$$\mathbf{B}(r, \theta, z, t) = \mathbf{B}(r, \theta)\, e^{ik_g z - i\omega t}$$

From the form of exponential $e^{ik_g z - i\omega t}$, it is apparent that

$$\partial/\partial z \to ik_g \quad \text{and} \quad \partial/\partial t \to -i\omega t$$

Substituting these values in (5), we get

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial E_z}{\partial \theta} - ik_g E_\theta &= i\omega B_r && \text{...(i)} \\ ik_g E_r - \frac{\partial E_r}{\partial r} &= i\omega B_\theta && \text{...(ii)} \\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r E_\theta) - \frac{\partial E_r}{\partial \theta}\right] &= i\omega B_z && \text{...(iii)} \\ \frac{1}{r}\frac{\partial B_z}{\partial \theta} - ik_g B_\theta &= \frac{i\omega}{c^2} E_r && \text{...(vi)} \\ ik_g B_r - \frac{\partial E_z}{\partial r} &= -\frac{i\omega}{c^2} E_\theta && \text{...(v)} \\ \frac{1}{r}\left[\frac{\partial}{\partial r}(r B_\theta) - \frac{\partial B_r}{\partial \theta}\right] &= -\frac{i\omega}{c^2} E_z && \text{...(vi)} \end{aligned}\right] \quad \text{...(6)}$$

Eliminating $B_\theta$ from (ii) and (iv), we get

$$E_r = \frac{l\omega}{\left[\frac{\omega^2}{c^2} - k_g^2\right]}\left[\frac{k_g}{\omega}\frac{\partial E_z}{\partial r} + \frac{1}{r}\frac{\partial B_z}{\partial \theta}\right] \quad \text{...(}$$

Eliminating $B_r$ from (i) and (v), we get

$$E_\theta = -\frac{i\omega}{\left[\frac{\omega^2}{c^2} - k_g^2\right]}\left[\frac{\partial B_z}{\partial r} - \frac{1}{r}\frac{k_g}{\omega}\frac{\partial E_z}{\partial \theta}\right]$$

Eliminating $E_\theta$ from (i) and (v), we get

$$B_r = + \frac{i}{\left[\frac{\omega^2}{c^2} - k_g^2\right]} \left[k_g \frac{\partial B_z}{\partial r} - \frac{\omega}{c^2 r} \frac{\partial E_z}{\partial \theta}\right] \qquad ...(9)$$

and finally eliminating $E_r$ from (ii) and (iv), we get

$$B_\theta = \frac{i}{\left[\frac{\omega^2}{c^2} - k_g^2\right]} \left[\frac{\omega}{c^2} \frac{\partial E_z}{\partial r} + \frac{k_g}{r} \frac{\partial B_z}{\partial \theta}\right] \qquad ...(10)$$

It is obvious that all the transverse components of field viz. $E_r$, $E_\theta$, $B_r$, $B_\theta$ may be obtained from $B_z$ and $E_z$. Thus we have to solve two dimensional wave equation (2) for $E_z$ and $B_z$. Let us now consider two usual cases :

**Case (i). TM Mode.** For *TM* mode $B_z = 0$ ; hence equations (7), (8), (9) and (10) take the form

$$E_r = \frac{ik_g}{\left[\frac{\omega^2}{c^2} - k_g^2\right]} \frac{\partial E_z}{\partial r} \qquad ...(a)$$

$$E_\theta = \frac{ik_g}{\left[\frac{\omega^2}{c^2} - k_g^2\right]} \frac{\partial E_z}{\partial \theta} \qquad ...(b)$$

$$B_r = - \frac{i\omega}{\left[\frac{\omega^2}{c^2} - k_g^2\right] c^2 r} \frac{\partial E_z}{\partial \theta} \qquad ...(c)$$

$$B_\theta = \frac{i\omega}{\left[\frac{\omega^2}{c^2} - k_g^2\right] c^2} \cdot \frac{\partial E_z}{\partial r} \qquad ...(d)$$

...(11)

Obviously in *TM* mode all transverse components are expressible in terms of $E_z$; hence we have to solve two dimensional wave equation for $E_z$ ; which is

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] E_z + k_g^2 E_z = 0 \qquad ...(12a)$$

with boundary condition $E_z = 0$ at $r = a$ for all values of $\theta$. ....(12b)

Let us solve above equation by the method separation of variables. Therefore writing

$$E_z (r, \theta) = R (r) \Theta (\theta) \qquad ...(13)$$

where $R (r)$ is a function of $r$ only, while $\Theta (\theta)$ is a function of $\theta$ only. Substituting $E_z = \Theta R$ in (12a) and dividing throughout by $R \Theta$, we get

$$\frac{1}{Rr}\frac{\partial}{\partial r}\left(r \frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} + k_c^2 = 0 \qquad ...(14)$$

Muliplying throughout by $r^2$, we get

$$\frac{r}{R}\frac{\partial}{\partial r}\left(\frac{\partial R}{\partial r}\right) + k_c^2 r^2 = -\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} \qquad ...(15$$

In above equation L.H.S. is a function of $r$ only ; while R.H.S. is a function of $\theta$ only. This is only possible if each side is equal to the same constant say $n^2$ i..e

$$-\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = n^2 \quad ...(16a)$$

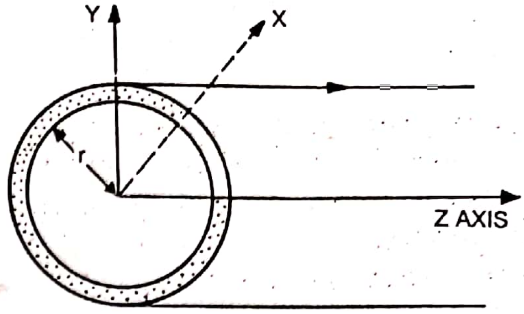


Fig. 9.13. Circular wave guide.

and
$$\frac{r}{R}\frac{r}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + k_c^2 r^2 = n^2 \quad ...(17)$$

Equation (16a) may be expressed as

$$\frac{\partial^2\Theta}{\partial\theta^2} + n^2\Theta = 0$$

the solution of equations (16b) is

$$\Theta = A_n \cos(n\theta) + B_n \sin(n\theta) \quad ...(18)$$

Further the relative amplitudes of $A_n$ and $B_n$ determine the orientation of the field in the guide and for a circular guide and for any particular values of $n$, the $\theta$-axes can always be oriented to make either $A_n$ or $B_n$ equal to zero. Let us choose $B_n$ equal to zero, so that

$$\Theta = A_n \cos n\theta$$

Now equation (17a) may be expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \left(k_c^2 - \frac{n^2}{r^2}\right)R = 0 \quad ...(19)$$

or
$$\frac{1}{r}\frac{\partial}{\partial(k_c r)}\left[r\frac{\partial R}{\partial(k_c r)}\right] + \left[1 - \frac{n^2}{(k_c r)^2}\right] = 0 \quad ...(20)$$

Equation (20) is a Bessel's equation in terms of $(k_c r)$ which has two independent solutions namely Bessel's function $J_n(k_c r)$ and Neumann's function $N_n(k_c r)$.

The Bessel's and Neumann's function [$J_n(x)$ and $N_n(x)$] have the following properties :

(i) At $x = 0$, the Bessel's function $J(x)$ is proportional to $x^n$ while Neumann's function $N_n(x)$ becomes infinite.

(ii) For large values of $x$, $J_n(x)$ and $N_n(x)$ approach the values

$$\left.\begin{aligned} J_n(x) &\rightarrow \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{(2n+1)}{4}\pi\right) \\ N_n(x) &\rightarrow \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{2n+1}{4}\pi\right)\end{aligned}\right] \quad ...(21)$$

From this it is clear that in a circular guide, where the field must be finite at $r = 0$, we must use only Bessel's function $J_n(k_c r)$. Thus the accepted solution of equation (10) is

$$R = J_n(k_c c) \quad ...(22)$$

Hence
$$E_z(k_c r) = R(r)\,.\,\Theta(\theta)$$
$$= J_n(k_c r) A_n \cos n\theta = A_n J_n(k_{cr}) \cos n\theta \quad ...(23a)$$

$\therefore$
$$E_z(r, \theta, z, t) = A_n J_n(k_c r)\cos n\theta\, e^{ik_g z - i\omega t} \quad ...(23b)$$

The boundary condition for $TM$ mode is

$$E_z\Big|_{r=a} = 0. \text{ This implies } J_n(k_c a) = 0 \quad ...(24)$$

Equation (24) has infinite number of roots but since for the propagation to be possible $k_g = \sqrt{(k_0^2 - k_c^2)}$ should be real *i.e.* $k_c$ should be small $k_c^2 \ll k_0^2$ or $k_c^2 \ll \omega^2/c^2$ or otherwise extremely high frequencies will be required therefore only first few roots of equation (24) will be of practical interest. The first few roots $J_n(x_{nm}) = 0$ are given below

$$\left.\begin{array}{l} x_{01} = 2{\cdot}405\,;\ x_{11} = 3{\cdot}832\,;\ x_{21} = 5{\cdot}135 \\ x_{01} = 5{\cdot}520\,;\ x_{12} = 7{\cdot}016\,;\ x_{22} = 8{\cdot}417 \end{array}\right\} \quad ...(25)$$

The various *TM* modes are represented by $TM_{01}$, $TM_{02}$, etc. *i.e.* in general $TM_{nm}$. It is obvious that since there is no root $x_{00}$ or $(k_c a)_{00}$. *$TM_{00}$ wave can not exist.*

Physically $n$ represented the number of cycles of variation of $E_z$ found as $\theta$ varies around the complete cylinder. The subscript $m$ indicates the number of zeros of electric field along a radial path from the centre to the inside surface of the guide wall. Thus if we let $x_{nm}$ be the $m$th value of $x$ for which $J_n(x) = 0$; then the cut off value of $k_c$ is given by

$$k_c a = x_{nm} \text{ or } k_c = \frac{x_{nm}}{a} \quad ...(26)$$

Hence *cut off wavelength* is given by

$$\lambda_c = \frac{2\pi}{k_c} = \frac{2\pi a}{x_{nm}} \quad ...(27)$$

the phase velocity is

$$v_g = \frac{\omega}{k_g} = \frac{\omega}{\sqrt{(k_0^2 - k_c^2)}} \quad ...(28a)$$

$$= \frac{\omega}{k_0 \sqrt{\left[1 - \left(\frac{k_c^2}{k_0^2}\right)\right]}}$$

$$= \frac{c}{\sqrt{\left[1 - \frac{k_c^2}{k_0^2}\right]}}$$

$$= \frac{c}{\sqrt{\left[1 - \frac{\omega_c^2}{\omega_0^2}\right]}} \quad ...(28b)$$

It is obvious that phase velocity is greater than the speed of light, and *group velocity* or *velocity of propagation of wave is*

$$v_z = \frac{\partial \omega}{\partial k_g} = \frac{\partial}{\partial k_g}(ck_o) = \frac{\partial}{\partial k_g}[c\,(k_g^2 + k_c^2)^{1/2}]$$

$$= \frac{ck_g}{\sqrt{(k_g^2 + k_c^2)}}$$

$$= \frac{ck_g}{k_0} = \frac{c^2 k_g}{\omega} \quad ...(29)$$

For a given $k_c$, this values varies from $o$ to $c$ when $k_c$ varies from $o$ to $\infty$.

Substituting solution for $E_z$ from (23b) and using

$$k_c^2 = k_0^2 - k_g^2 = (\omega^2/c^2) - k_g^2$$

the *transverse field components* may be expressed from equation (11) *viz.*

$$\left.\begin{aligned} E_r &= \frac{ik_g}{k_c^2} A_n J_n'(k_c r) \cos n\theta \, e^{ik_g z - i\omega t} && \text{...(a)} \\ E_\theta &= -\frac{ik_g n}{k_c^2 r} A_n J_n(k_c r) \sin n\theta \, e^{ik_g z - i\omega t} && \text{...(b)} \\ B_r &= +\frac{i\omega n}{k_c^2 c^2 r} A_n J_n(k_c r) \sin n\theta \, e^{ik_g z - i\omega t} && \text{...(c)} \\ B_\theta &= \frac{i\omega}{k_c^2 c^2} A_n J_n'(k_c r) \cos n\theta \, e^{ik_g z - i\omega t} && \text{...(d)} \end{aligned}\right\} \quad \text{...(30)}$$

where $J_n'$ represents derivative of $J_n$ with respect to $r$.

**Case (ii).** ***TE* Mode.** For *TE* mode $E_2 = 0$ ; hence equations (7), (8), (9) and (10) take the form

$$\left.\begin{aligned} E_r &= \frac{i\omega}{[(\omega^2/c^2) - k_g^2]_r} \frac{\partial B_z}{\partial \theta} && \text{...(a)} \\ E_\theta &= \frac{i\omega}{[(\omega^2/c^2) - k_g^2]} \frac{\partial B_z}{\partial r} && \text{...(b)} \\ B_r &= \frac{ik_g}{[(\omega^2/c^2) - k_g^2]} \frac{\partial B_z}{\partial r} && \text{...(c)} \\ B_\theta &= \frac{ik_g}{[(\omega^2/c^2) - k_g^2]} \frac{\partial B_z}{\partial \theta} && \text{...(d)} \end{aligned}\right\} \quad \text{...(31)}$$

Obviously in *TE* mode all transverse components $E_r$, $E_\theta$, $B_r$, $B_\theta$ are expressible in terms of $B_z$, hence we have to obtain the solution for two dimensional wave equation (1) for $B_z$. The solution is obtained exactly as in *TM* mode we get

$$B_z(r, \theta) = A_n J_n(k_c r) \cos n\theta \quad \text{...(32)}$$

The boundary condition for *TE* mode is $\partial B_z/\partial r = 0$ at $r = a$ for all values of $\theta$.

This implies

$$\left.\frac{\partial J_n(k_c r)}{\partial r}\right|_{r=a} = 0 \text{ or } J'_n(k_c a) = 0 \quad \text{...(33)}$$

The derivative $J_n'(k_c r)$ with respect to $r$ of the Bessel function may be obtained from

$$J'_n(k_c r) = \frac{\partial J(k_c r)}{\partial r} = \frac{n}{k_c r} J_n(k_c r) - J_{n+1}(k_c r) \quad \text{...(34}$$

The first few roots of $J'_n(x_{nm}) = 0$ are

$$\left.\begin{aligned} x_{01} &= 3{\cdot}832, & x_{11} &= 1{\cdot}842, & x_{21} &= 3{\cdot}05 \\ x_{02} &= 7{\cdot}016, & x_{12} &= 5{\cdot}330, & x_{22} &= 6{\cdot}71 \end{aligned}\right\} \quad \text{...(3}$$

If we let $x_{nm}$ by the $m^{th}$ value of $x$ for which $J_n'(x) = 0$, then cut off value of $k_c$ is given by

$$k_c a = x_{nm} \text{ or } k_c = \frac{x_{nm}}{a}. \quad \text{...(3}$$

Hence *cut off wavelength* $$\lambda_c = \frac{2\pi}{k_c} = \frac{2\pi a}{x_{nm}} \quad ...(36b)$$

where $x_{nm}$ are given by equation (35). Thus we can calculate the cut off wavelength of various modes. The equations for phase velocity and group velocity are the same as those obtained for *TM* mode except for the use of equation (35) for $x_{nm}$ in place of equation (25).

The field components for *TE* mode may by expressed as

$$\left.\begin{aligned} E_z &= 0 \text{ and } B_z = A_n J_n (k_c r) \cos n\theta\, e^{ik_g z - i\omega t} \\ E_r &= -\frac{i\omega n}{k_c^2 r} A_n J_n (k_c r) \sin n\theta\, e^{ik_g z - i\omega t} \\ E_\theta &= -\frac{i\omega}{k_c^2} A_n J'_n (k_c r) \cos n\theta\, e^{ik_g z - i\omega t} \\ B_r &= \frac{ik_g}{k_c^2} A_n J'_n (k_c r) \cos n\theta\, e^{k_g z - i\omega t} \\ B_\theta &= -\frac{ik_g n}{k_c^2 r} A_n J_n (k_c r) \sin n\theta\, e^{ik_g z - i\omega t} \end{aligned}\right\} \quad ...(37)$$

## 9.15. Resonant Cavities

If the cylindrical wave guide of finite length has its end faces then it forms what is known as *resonant cavity* (or cavity resonator). Such cylindrical cavity resonator may be rectangular or circular cavity resonator depending upon whether the cross-section of the cavity resonator is rectangular of circular. We shall restrict ourselves to case where cross-section is rectangular. We shall assume that :

(i) the end faces are plane and perpendicular to the axis of cavity.

(ii) the walls are perfectly conducting.

(iii) the interior of cavity is vacuum.

An electromagnetic wave propagating along the axis of a cavity will be reflected back and forth from the end faces, thus giving rise to *standing wave pattern.* If *r* is the amplitude of reflection coefficient, then the amplitude of transverse electric field is *r* times of the amplitude of incident wave ; consequently the amplitude of resulting disturbance, obtained by superposition of incident and reflected waves, will vary from something proportional to $(1 + r)$ ; where the two waves add, to something proportional to $(1 - r)$ ; where they oppose each other. The ratio of maximum to minimum *E* or $\frac{1+r}{1-r}$ is called the *standing wave ratio* in voltage. We see that it can go to unity $r = 0$ when there is no reflection to $r = 1$ when there is perfect reflection. For metallic reflection which we shall assume $r = 1$, so that the *standing wave ratio is infinite.* The points along the guide where the resultant transverse electric field is maximum are called *standing wave maxima* and those where it is minimum are called *standing wave minima.* For in infinite standing wave ratio the standing wave minima become the nodes and in any case they are separated by a half wave length.

It is observed that certain standing wave field distributions be set up in the cavity which correspond to different cavity modes. The possible frequencies are known as resonant frequencies of the cavity. Consider a rectangular cavity consisting of the region bounded by the six planes $x = 0, x = a, y = 0, y = b, z = 0, z = d.$

**Case (i). *TE* Mode.** In this mode $E_x = 0$ ; let us put

$$\mathbf{E} = \mathbf{E}_\perp + \mathbf{E}_z$$

where $$\mathbf{E}_\perp = (\mathbf{i}\, E_x + \mathbf{j}\, E_y) \quad ...($$

Solve the problems :-

1. What must be the width of a rectangular guide such that the energy of electromagnetic radiation whose free space wave length is 3.0 cm, travels down the guide at 95 % of the speed of light?

2. For transverse electric waves perfectly propagating in a rectangular wave guide with perfectly conducting walls; find

(i) the cut off wave length.
(ii) the magnetic field induction.
(iii) The velocity with which energy is transmitted along the guide.

Reference :=

· Satya Prakash [Electromagnetic theory and Electrodynamics]