- SEMESTER –II (HONOURS)
- PAPER : CC4T
- TOPIC: VIBRATION OF STRING

CLASS NOTES :BY TAPAS KUMAR CHANDA

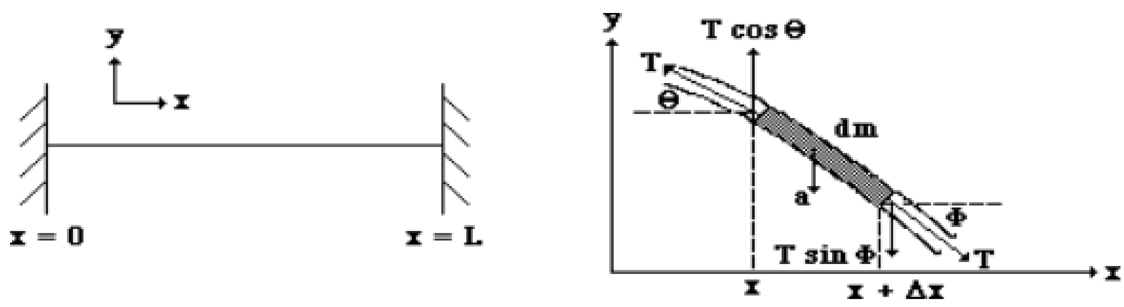
**Dept. of physics, Bhatter College,Dantan**

**Sub topics: solution of transverse wave equation ,plucked string.**

Date: 05/04/2020

## Equation of transverse wave in a string:

Consider a thin, flexible string (piano wire, rope, etc.) of length L, linear mass density $\mu$, under tension T, which is fixed at both ends as shown in figure 1. Two questions we might ask are whether waves can exist in such a system and if so what is the form of the function y(x,t) which describes the propagation of the wave?




(a) (b)

Figure 1

If a system will support waves, then the equation describing the behavior of the system will have the form of the classical wave equation,

$$\frac{\partial^2 y}{\partial x^2} - 1/v^2 \frac{\partial^2 y}{\partial t^2} = 0 \qquad . \qquad (1)$$

Therefore, to answer the first question posed above, we need to derive the equation of motion for our string and compare its form to that of equation (1). To answer the second question, we need to look at the effect the fixed ends have on waves traveling down the string. In what follows we will answer these questions.

Consider a small section of the string dm which has been displaced in the vertical direction as shown in figure 1a. The displacement is y = y(x,t). We will assume that the displacement is small and that $\theta$ and $\phi$ are everywhere small so that we can use the approximations $\cos\theta \approx \cos\phi \approx 1$, $\theta \approx \sin\theta \approx \tan\theta$ and $\phi \approx \sin\phi \approx \tan\phi$. The element dm is acted upon by two forces, the tension T at both ends. (Since the string is thin, gravitational forces can be neglected). The forces in the horizontal and vertical directions are

$$F_x = T\cos\phi - T\cos\theta \qquad (2a)$$

$$F_y = T\sin\theta - T\sin\phi \qquad (2b)$$

Since $\cos\theta \approx \cos\phi \approx 1$, the horizontal forces cancel leaving a net force only in the y direction. Applying the small angle approximation to equation (2b) yields

$$F_y = T(\tan\theta - \tan\phi) \quad . \quad (3)$$

But $\tan\theta = -\partial y/\partial x|_x$ and $\tan\phi = -\partial y/\partial x|_{x+\Delta x}$. Substituting these expressions into equation (3) gives

$$F_y = T\left(\left(-\frac{\partial y}{\partial x}\right)_x - \left(-\frac{\partial y}{\partial x}\right)_{x+\Delta x}\right) \quad . \quad (4)$$

To make the notation simpler, we define a function $g(x) = \partial y/\partial x|_x$. Substituting this into equation (4) and rearranging terms yields

$$F_y = T[g(x+\Delta x) - g(x)] \quad . \quad (5)$$

Applying Newton's second law gives

$$ma_y = T[g(x+\Delta x) - g(x)] \quad . \quad (6)$$

But $m = \mu\Delta x$ and $a_y = \partial^2 y/\partial t^2$. Substituting these expressions into equation (6) and rearranging terms yields

$$\frac{g(x+\Delta x) - g(x)}{\Delta x} - \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = 0 \quad . \quad (7)$$

Realizing that $\frac{g(x+\Delta x) - g(x)}{\Delta x} = \frac{\partial g(x)}{\partial x} = \frac{\partial^2 y}{\partial x^2}$, allows us to rewrite equation (7) in its final form,

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = 0 \quad . \quad (8)$$

If we let $\mu/T = 1/v^2$, then we see that the equation governing the motion of the string has the same form as the classical wave equation. Therefore, waves can exist in our system. The waves will travel with velocity $v = (T/\mu)^{1/2}$ and the function y(x,t) will be numerically equal to the y displacement at a time t of a point on the string at position x.

Now that we know that waves can exist in our system, we can turn our attention to the question of the form of the function y (x,t). The fact that the ends are fixed means that the y amplitude must always be zero at the ends. Therefore, only those functions for which y (0,t) = 0 = y (L,t) are suitable solutions. It can be shown by substitution that functions of the form $y = A \sin(Kx \pm \omega t)$ and $y = B\cos(Kx \pm \omega t)$ [where K(wave number) and $2\pi/\lambda$ and $\omega$ (angular frequency) = $2\pi\nu$ = v K], are solutions to equation (8). However, since the functions y = Bcos

$(Kx \pm \omega t)$ cannot always be zero when $x = 0$ we can eliminate that set of functions. To determine the form of the function y (x,t) for our system we must use waves of the form $y = A \sin (Kx \pm \omega t)$.

The function $y_1 = A \sin (Kx - \omega\tau)$ represents a continuous sine wave traveling to the right down the string, and $y_2 = A \sin (Kx + \omega t)$ represents one traveling to the left. If these waves are perfectly reflected at the ends, we have two waves of equal frequency, amplitude and speed traveling in opposite directions on the same string. The principle of superposition of waves states that the resulting wave will be the algebraic sum of the individual waves,

$$y = y_1 + y_2 = A[\sin(Kx - \omega t) + \sin(Kx + \omega t)] \quad ;$$

or using the trigonometric identity for the sum of the sines of two angles ($\sin B + \sin C = 2 \sin 1/2(B+C) \cos 1/2(C-B)$), we obtain

$$y = 2A \sin Kx \cos \omega t \quad . \quad (9)$$

This function obviously satisfies the boundary conditions at $x = 0$, but will only satisfy the boundary condition at $x = L$ when $K = n\pi/L$ (where n = 1,2,3...). Limiting the values of K to only certain values also limits the wavelength, frequency and speed of the waves to certain discrete values. Therefore, unlike traveling waves on an infinite string which can have any wave- length or frequency, waves on a bounded string are quantized, restricted to only certain wavelengths and frequencies. To note this quantization, equation (9) can be rewritten as

$$y_n = 2 A_n \sin K_n x \cos \omega_n t \quad (10)$$

Where $K_n = n\pi/L$, $\omega_n = K_n v = \frac{n\pi}{L}\sqrt{T/\mu}$ and $n = 1, 2, 3, \cdots$ .

Equation (10) is the equation of a standing wave. Note that a particle at any particular point x executes simple harmonic motion as time passes and that all particles vibrate with the same frequency. Note also that the amplitude is not the same for different particles, but varies with the location x of the particle. The amplitude, $2 A_n \sin K_n x$, has a maximum value of $2A_n$ at positions where $Kx = \pi/2, 3\pi/2, 5\pi/2$ etc. or where $x = \lambda/4, 3\lambda/4, 5\lambda/4$, etc. These points are called antinodes and are spaced one half wavelength apart. The amplitude has a minimum value of zero at positions where $Kx= \pi, 2\pi, 3\pi$, etc. or $x = \lambda/2, , 3\lambda/2, 2\lambda,$ etc. These points are called nodes and are also spaced one half wavelength apart.

Finally, it should be noted that although equation (10) is a form of wave which can exist in the bounded string system, it is not the most general form. The most general form is

$$y_n(x, t) = \sum_{n=1}^{\alpha} (a_n \cos \omega_n t + B_n \sin \omega_n t) \quad . \quad (11)$$

Now that we have determined that waves can exist in our system and how they can be represented mathematically, we might ask what we would expect to see if we tried to create the waves in an actual string. Consider then a string fixed at both ends which is being driven by a force F cos $\omega t$. If the driving frequency is such that the distance L between the ends is neither an integral or half-integral number of wavelengths, the initial and reflected waves will be "out of phase" and will destructively interfere with each other. No clear pattern will be set up. If however the string is driven with a frequency near $\omega_n$ so that L is an integral or half-integral number of wavelengths, the initial and reflected waves will be "in phase" and will constructively interfere. The standing wave $y_n(x,t)$ will be produced and will attain a large amplitude. If n = 1 then $L = \lambda/2$ and the string is said to be vibrating at its fundamental frequency. This is the lowest frequency for which a standing wave pattern can be set up in the string. If the string is driven at a frequency which an integral multiple of the fundamental frequency, standing waves with different patterns will be set up. The patterns for the first four frequencies are shown in figure 2.

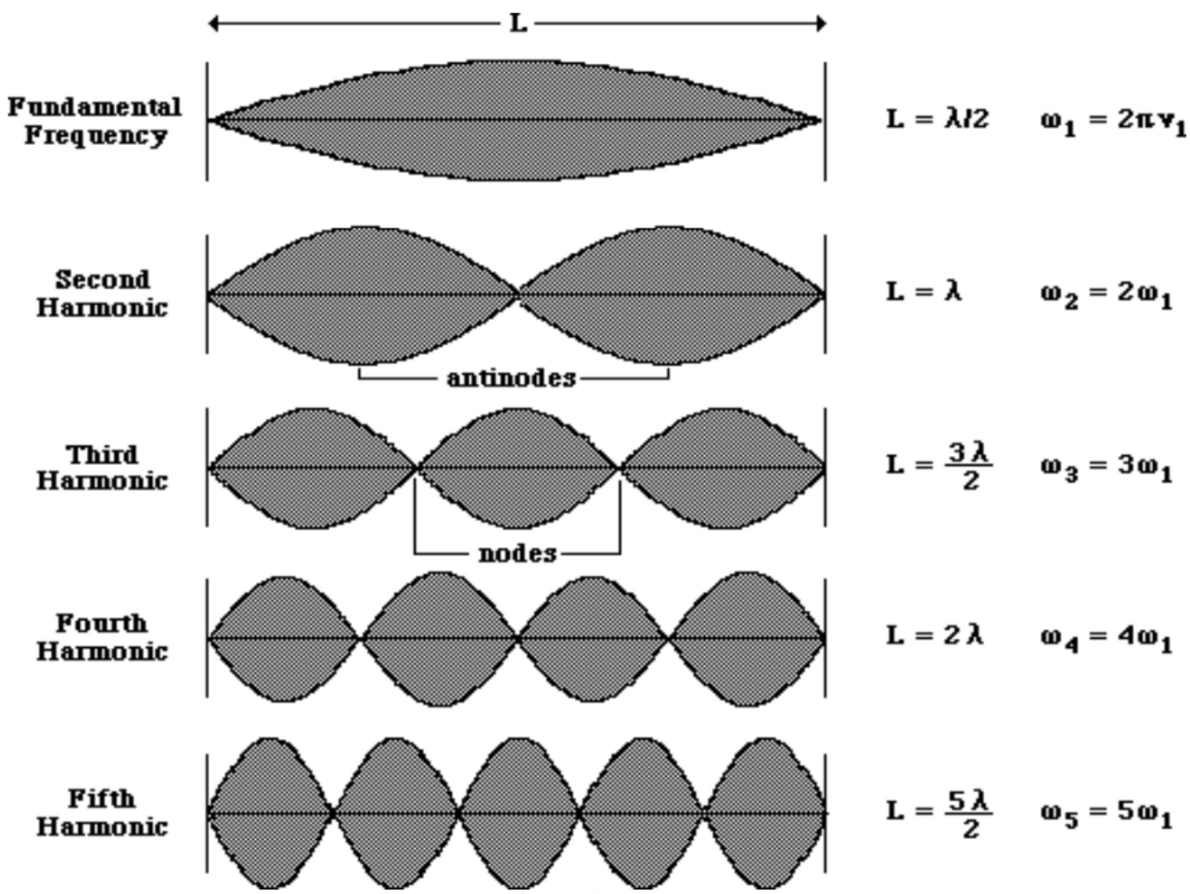



Figure 2

When the string is driven at one of its natural frequencies and its amplitude is near maximum it is said to be in resonance. A plot of $|y^2|$ vs. $\omega$ is shown in figure 3. Such a graph is called a resonance curve. Near resonance the energy transfer (from driving mechanism to string is at its most efficient level.

Finally it should be noted that if the string is plucked rather than driven by a periodic force, then in general the response y(x,t) will not be a single natural frequency but a sum of many natural frequencies.

$$y_n(x, t) = \sum_n \left(A_n \cos \omega_n t + B_n \sin \omega_n t\right) \sin K_n x \quad . \quad (11)$$

The observed pattern is very complicated in general. However, it is possible to pluck the string so as to have one natural frequency dominate.

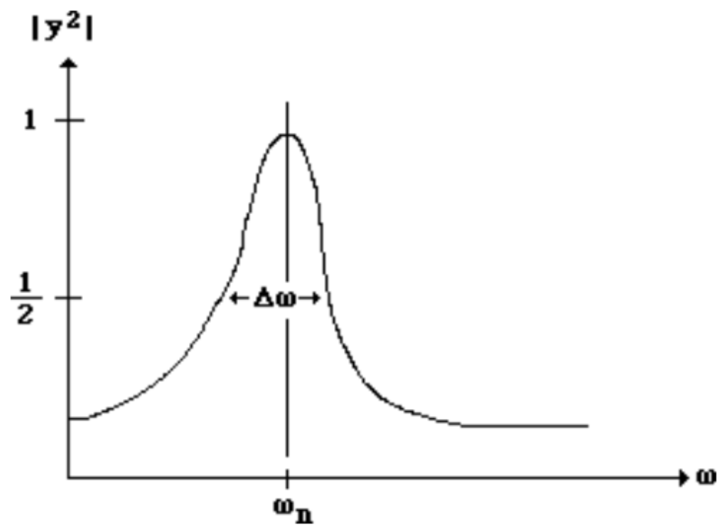



Figure 3

## 4. Solving the 1D Homogeneous Wave Equation with Separation of Variables

We now want to solve the wave equation in 1 spatial dimension (1-D), equation (17). This equation governs wave propagation in a 1-D medium, such as a string or a wire.

Partial differential equations such as equation (17) are usually not solved directly, but are transformed into other equations that can be solved. Usually they are transformed first into a set of ODEs, one for each free variable. For the 1-D wave equation, therefore, we'll expect two equations, one in $x$ and one in $t$. The method we're going to follow now is called the *method of separation of variables.*

Equation (17) can be separated into these two constitutive equations by using the method of separation of variables in the following way. Let us assume that the solution can be written (as we know it can for a string) in terms of the product of two functions, one in $x$ and the other in $t$, in the following way:

$$y(x,t) = Y(x)T(t) \tag{20}$$

$Y(x)$ and $T(t)$ are the unknowns we wish to find and equation (20) is a a kind of trial solution and we'll see if it works. To substitute equation (20) into equation (17) we'll first need the space and time derivatives of $y$:

$$\begin{aligned}
\frac{\partial y(x,t)}{\partial x} &= T(t)\frac{\partial Y(x)}{\partial x} = T(t)\frac{dY(x)}{dx} \\
\frac{\partial^2 y(x,t)}{\partial x^2} &= T(t)\frac{\partial^2 Y(x)}{\partial x^2} = T(t)\frac{d^2Y(x)}{dx^2} \qquad (21)\\
\frac{\partial y(x,t)}{\partial t} &= Y(x)\frac{\partial T(t)}{\partial t} = Y(x)\frac{dT(t)}{dt} \\
\frac{\partial^2 y(x,t)}{\partial t^2} &= Y(x)\frac{\partial^2 T(t)}{\partial t^2} = Y(x)\frac{d^2T(t)}{dt^2} \qquad (22)
\end{aligned}$$

Note that we've replaced the partial derivatives on the right-hand side with total derivatives because they are derivatives of functions of a single variable. Substituting equations (21) and (22) into equation (17) we get:

$$\frac{d^2T(t)}{dt^2}Y(x) = c^2\frac{d^2Y(x)}{dx^2}T(t)$$

which upon rearranging yields:

$$\frac{1}{c^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = \frac{1}{Y(x)}\frac{d^2Y(x)}{dx^2} \tag{23}$$

Note that the left-hand side of equation (23) is just a function of $t$ and the right-hand side is only a function of $x$.

**Now, comes the key step.** It's simple, but you have to pay attention. How can a function of $t$, which in principle could be changing arbitrarily in time, be equal to a function of $x$ that may be changing arbitrarily in space? Well, to make a long story short, the only way is if both sides of equation (23) are equal to the same constant which is called the *separation constant.* For a reason that will become apparent later, let's let that constant be called $-k^2$, so:

$$\begin{aligned}
\frac{1}{c^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} &= -k^2 \\
\frac{1}{Y(x)}\frac{d^2Y(x)}{dx^2} &= -k^2
\end{aligned}$$

which after a little rearranging can be rewritten as:

$$\frac{d^2Y(x)}{dx^2} + k^2Y(x) = 0 \tag{24}$$

$$\frac{d^2T(t)}{dt^2} + c^2k^2T(t) = 0 \Rightarrow \frac{d^2T(t)}{dt^2} + \omega^2T(t) = 0 \tag{25}$$

where the latter result in equation (25) holds because $\omega = ck$.

Equations (24) and (25) are the two ODEs whose solutions, $Y(x)$ and $T(t)$, can be substituted into equation (20) to give a solution to the PDE, the wave equation given by equation (17). Comparison of equations (24) and (25) with equation (6) reveals that both of these equations are simply Helmholtz equations, which we know how to solve because of their role in the SHO. Their solutions, therefore, are simply:

$$Y(x) = A\cos kx + B\sin kx \tag{26}$$

$$T(t) = C\cos\omega t + D\sin\omega t \tag{27}$$

where $A, B, C$, and $D$ are arbitrary constants. You can see why we defined the separation constant as $-k^2$ because doing so yields equation (26) where $k$ plays the role of wavenumber as we have defined it previously.

The boundary conditions allow us to find $A$ as well as $k$ and, hence, $\omega$ as we will now show. The initial conditions will specify the products $BC$ and $BD$. This is discussed further in the next section.

Now, let's apply the boundary conditions. Assume that the string is clamped both at both ends: $x = 0$ and $x = a$. The boundary conditions, therefore, are $y(0,t) = y(a,t) = 0$ or equivalently $Y(0) = Y(a) = 0$, so using equations (26) and (27) we see that:

$$0 = Y(0) = A\cos(0) + B\sin(0) \Rightarrow A = 0 \tag{28}$$

$$0 = Y(a) = B\sin ka \Rightarrow k = \frac{1}{a}\sin^{-1}(0) \Rightarrow k_n = \frac{n\pi}{a}, \tag{29}$$

where $n$ is an integer. Remember that the expression $\sin^{-1}(0)$ should be read as the angle(s) at which sine is zero; which is just multiples of $\pi$.

We see, therefore, that we've established that there are a countably infinite number of allowable separation constants $k$ indexed by the number $n$, that we recognize as the mode number or quantum number as discussed above. In section 1, we established that $k_n = n\pi/a$ based on purely physical considerations, here the reasoning was more mathematical but the result is the same. We see now that:

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{a} \qquad \omega_n = ck_n = \frac{n\pi c}{a}, \tag{30}$$

which is the same as equations (10) above. You can see through equations (28) and (29) how the boundary conditions determine the frequencies of oscillation in practice.

The final solution $y(x,t)$ is a linear combination of all of the solutions indexed by $n$:

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} Y_n(x)T_n(t) = \sum_{n=1}^{\infty} B_n \sin k_n x\,(C_n\cos\omega_n t + D_n\sin\omega_n t) \tag{31}$$

$$= \sum_{n=1}^{\infty} \sin k_n x\,(A'_n\cos\omega_n t + B'_n\sin\omega_n t) = \sum_{n=1}^{\infty} C'_n \sin k_n x\,(\sin(\omega_n t - \phi_n)) \tag{32}$$

where we recombined the three arbitrary constants into two ($A'_n \equiv B_n C_n$ and $B'_n \equiv B_n D_n$) and also rewritten in terms of a phase shift $\phi_n$ which we will reference in the discussion of energy below. This reproduces the physically motivated equation (5) above. As before, the initial conditions will determine the coefficients ($A'_n$, $B'_n$) or ($C'_n$, $\phi_n$).

**5. Application of Initial Conditions**

For a string clamped at both ends, the the solution for displacement $y(x,t)$, dropping the primes on the coefficients, is:

$$y(x,t) = \sum_{n=1}^{\infty} \sin k_n x \left(A_n \cos \omega t + B_n \sin \omega t\right), \tag{33}$$

where the coefficients $A_n$ and $B_n$ depend on how the string is set into motion, i.e., on the initial conditions, and $k_n = n\pi/L$ and $\omega_n = ck_n$ where $L$ is the length of the string and $c$ is the speed of propagation of waves on the string.

If $f(x)$ and $g(x)$ are the initial patterns of displacement and velocity imparted to the string, then from equation (33) we see that:

$$y(x,0) \quad = \quad f(x) = \sum_{n=1}^{\infty} A_n \sin k_n x = \sum_{n=1}^{\infty} a_n \sin k_n x, \tag{34}$$

$$v(x,0) = \dot{y}(x,0) \quad = \quad g(x) = \sum_{n=1}^{\infty} \omega_n B_n \sin k_n x = \sum_{n=1}^{\infty} b_n \sin k_n x. \tag{35}$$

The final equality in equations (34) and (35) is just the expansion of $f(x)$ and $g(x)$ in a Fourier Series. In both cases, the Fourier Series is only a sine-series because the boundary conditions require that the function go to zero at the end-points ($x = 0, x = L$). As usual, the coefficients in the Fourier Series are given by:

$$a_n = \frac{2}{L}\int_0^L f(x)\sin(k_n x)dx, \tag{36}$$

$$b_n = \frac{2}{L}\int_0^L g(x)\sin(k_n x)dx, \tag{37}$$

Here the constant in front of the integral is $2/L$ rather than $1/L$ because of interval we're considering goes from 0 to $L$ rather than $-L/2$ to $L/2$. Comparison of equations (34) and (35) with (36) and (37) reveals that:

$$A_n \quad = \quad a_n = \frac{2}{L}\int_0^L f(x)\sin(k_n x)dx, \tag{38}$$

$$B_n \quad = \quad \frac{b_n}{\omega_n} = \frac{2}{\omega_n L}\int_0^L g(x)\sin(k_n x)dx. \tag{39}$$

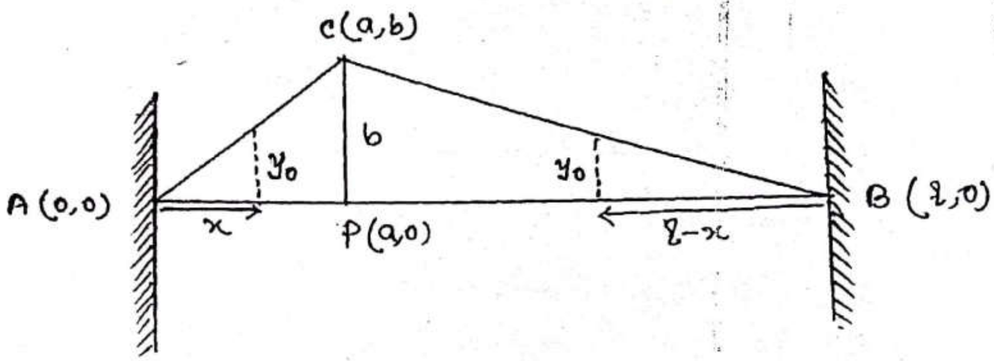
These equations together with equation (33) give the solution to the problem with the initial conditions imposed.

## (1) Plucked String *

Remember

If a string is fixed at its two ends and the string is plucked at a point and pulled to a vertical distance and then released; then the vibration will be set up, then the string is called plucked string.

Example —— Guitar, sitar, tanpura ..




Let the string AB is fixed at the points A(0,0) and B(l,0) along x-direction. Initially it is plucked at P(a,0) to C(a,b).

Boundary conditions, i) $y(x,0) = y_0$.

ii) $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$.

For the region $0 \le x \le a$, $\frac{y_0}{b} = \frac{x}{a}$ i.e. $y_0 = \frac{bx}{a}$.

For the region $a \le x \le l$, $\frac{y_0}{b} = \frac{l-x}{l-a}$ i.e. $y_0 = \frac{b(l-x)}{l-a}$.

General equation of string,

$$y(x,t) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi vt}{l} + B_n \sin \frac{n\pi vt}{l} \right) \sin \frac{n\pi x}{l} \quad \text{—(1)}$$

$$\therefore \frac{dy}{dt} = \sum_{n=1}^{\infty} \frac{n\pi v}{l} \left( -A_n \sin \frac{n\pi vt}{l} + B_n \cos \frac{n\pi vt}{l} \right) \sin \frac{n\pi x}{l} \quad \text{—(2)}$$

Putting b.c. ii) in equation (2),

$$0 = \sum_{n=1}^{\infty} \frac{n\pi v}{l} \times B_n \times \sin \frac{n\pi x}{l}$$

$$\therefore B_n = 0 \quad \text{as } \sin \frac{n\pi x}{l} \neq 0.$$

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi v t}{l} \quad \text{———— (3)}$$

Putting b.c. i) in equation (3),

$$y_0 = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} \quad \text{———— (4)}$$

To evaluate $A_n$, multiply both sides by $\sin\frac{m\pi x}{l}$ and integrate w.r.to $x$ from $x=0$ to $x=l$.

i.e, $$\int_{x=0}^{l} y_0 \sin\frac{m\pi x}{l}\,dx = \sum_{n=1}^{\infty} A_n \int_{x=0}^{l} \sin\frac{n\pi x}{l}\cdot\sin\frac{m\pi x}{l}\,dx$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = l/2$ when $n=m$

we get only one terms, when $n=m$,

$$\therefore \quad A_n = \frac{2}{l}\int_{x=0}^{l} y_0 \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^a \frac{bx}{a}\cdot\sin\frac{n\pi x}{l}\,dx + \int_a^l \frac{b(l-x)}{l-a}\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2b}{la}\left[-\frac{x\cos n\pi x/l}{n\pi/l} + \frac{\sin n\pi x/l}{(n\pi/l)^2}\right]_0^a$$

$$+ \frac{2b}{l(l-a)}\cdot l\cdot\left[-\frac{\cos n\pi x/l}{n\pi/l}\right]_a^l - \frac{2b}{l(l-a)}\left[-\frac{x\cos n\pi x/l}{n\pi/l} + \frac{\sin n\pi x/l}{(n\pi/l)^2}\right]_a^l$$

$$= \frac{2b}{l}\left[-\frac{a\cos n\pi a/l}{a\, n\pi/l} + \frac{\sin n\pi a/l}{a\,(n\pi/l)^2} + \frac{l}{(l-a)}\frac{\cos n\pi}{n\pi/l} + \frac{l}{l-a}\cdot\frac{\cos n\pi a/l}{n\pi/l}\right.$$

$$\left. + \frac{l\cos n\pi}{(l-a)\,n\pi/l} - \frac{\sin n\pi}{(l-a)\,(n\pi/l)^2} - \frac{a\cos n\pi a/l}{(l-a)\,n\pi/l} + \frac{\sin n\pi a/l}{(l-a)\,(\frac{n\pi}{l})^2}\right]$$

$$= \frac{2b}{l}\left[-\frac{l}{n\pi} + \frac{l^2}{n\pi(l-a)} - \frac{al}{n\pi(l-a)}\right]\cos\frac{n\pi a}{l}$$

$$+ \frac{2b}{l}\left[\frac{l^2}{n^2\pi^2 a} + \frac{l^2}{n^2\pi^2(l-a)}\right]\sin\frac{n\pi a}{l}.$$

$$= \frac{2bl^2}{n^2\pi^2 a(l-a)}\cdot\sin\frac{n\pi a}{l}.$$

Hence the resultant solution,

$$y(x,t) = \frac{2bl^2}{\pi^2 a(l-a)} \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \sin\frac{n\pi a}{l} \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi v t}{l}$$

## Young ~ Helmholtz law :—

For plucked string, the amplitude of nth harmonic,

$$S_n = \frac{2bl^2}{\pi^2 a(l-a)} \cdot \frac{1}{n^2} \sin\frac{n\pi a}{l} \sin\frac{n\pi x}{l}.$$

nth harmonic will be absent if $S_n = 0$

i.e, $\sin\frac{n\pi a}{l} = 0 = \sin m\pi \quad ; m = 1, 2, 3 \cdots$

or, $\frac{n\pi a}{l} = m\pi$

or, $n = \frac{ml}{a}$

i) If the string is plucked at mid-point then $a = l/2$ then $n = 2m$
i.e, 2nd, 4th, 6th ---------- harmonics will be absent.

ii) If the string is plucked at $a = l/3$ then $n = 3m$.
i.e, 3rd, 6th, 9th - ------- harmonics will be absent.

iii) .